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On Four-Dimensional Compactifications of F-Theory

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ABSTRACT

Branches of moduli space of F-theory in four dimensions are investigated. The transition between two branches is described as a 3-brane–instanton transition on a 7-brane. A dual heterotic picture of the transition is presented and the F-theory — heterotic theory map is given. The F-theory data — complex structure of the Calabi-Yau fourfold and the instanton bundle on the 7-brane is mapped to the heterotic bundle on the elliptic Calabi-Yau threefold CY_3. The full moduli space has a web structure which is also found in the moduli space of semi-stable bundles on CY_3. Matter content of the four-dimensional theory is discussed in both F-theory and heterotic theory descriptions.

1. Introduction

Great progress has been made recently in our understanding of six-dimensional compactifications of F-theory on elliptic Calabi-Yau threefolds [1]. The structure of the singular locus of elliptic fibration encodes the information about both the enhanced gauge symmetries and the matter contents of F-theory compactification [1][2]. F-theory also provides us with a powerful tool in studying the nonperturbative aspects of heterotic string compactifications, in the case when the heterotic dual exists.

In this paper we focus mainly on four-dimensional compactifications of F-theory. Four-dimensional compactifications appear to be very different from six-dimensional ones. First of all, these compactifications generically have a 3-brane anomaly [3][4][5]. The RR 4-form has an uncompensated 3-brane charge

$$\alpha = \frac{1}{24}\chi(CY_4), \tag{1.1}$$

where χ is the Euler number of Calabi-Yau fourfold. In order to cancel this anomaly one can insert an appropriate number of 3-branes [1]. When F-theory has a heterotic dual these 3-branes should correspond to the heterotic 5-branes [7][8]. Therefore, one novel feature is that we have to learn how to deal with 3-(5-) branes.

Another novelty is that on the compact part of the world-volume of the 7-brane, one can turn on the gauge field background [9] with a nonzero instanton number. Only when the background is trivial the four-dimensional gauge group is the one prescribed by the singularities of the elliptic fibration. Any nontrivial background breaks the gauge group to a smaller one. Also, in the presence of nontrivial background, the anomaly counting (1.1) changes: $\alpha = \chi/24 - k$, where k is the total number of instantons inside the 7-branes.

The properties of the four-dimensional N=1 supersymmetric field theories are determined by the configuration of 7-branes and 3-branes that intersect over a common flat $R^{3,1}$. The gauge groups come from both 7-branes and 3-branes. Many ways to distribute the anomaly α between 3-branes and instantons give rise to many branches of N=1 four-dimensional theory. These branches and transitions between them have a nice interpretation in terms of D-brane physics. For example, a single 3-brane produces a $U(1)$ factor in the full gauge group of the four-dimensional theory. It also contributes by 1 to cancellation of α. The position of the 3-branes inside the base of the elliptic fibration parameterizes the moduli space of the $U(1)$ theory. In particular it determines [8][10] the masses of the chiral superfields coming from strings connecting the 3-brane with 7-branes. When the 3-brane approaches a 7-brane, some of these fields become massless. At this very moment a transition to the Higgs branch becomes possible *if* some conditions are satisfied. On the Higgs branch a 3-brane "dissolves" into a finite size instanton of the nonabelian gauge group [11][12] and the 3-brane $U(1)$ gauge group disappears. If the appropriate conditions

[1] In this paper we will not discuss effects of possible nontrivial discrete background 3-brane fluxes [6].

are not satisfied, a superpotential gets generated that prevents a theory from developing the Higgs branch.

The pure Higgs branch corresponds to the situation when all 3-branes are replaced by the instantons and the anomaly is cancelled by the nonabelian gauge field. The mixed branches are the ones with both 3-branes and instantons.

If an F-theory compactification has a heterotic dual, this variety of branches finds its counterpart in the variety of branches of the moduli space of bundles on Calabi-Yau threefolds. In this paper we mainly consider the $SU(n)$ vector bundles. Very much unlike the situation with bundles on $K3$, the moduli space $\mathcal{M}_{CY_3}(n, c_2, c_3)$ of bundles with the fixed rank and Chern classes can have many irreducible components with different dimensions. To describe bundles on *elliptic* Calabi-Yau's we will use a very useful tool — the theory of spectral covers. This mathematical construction is well-known in the context of Hitchin systems [13]. Its applications to the heterotic string compactifications are developed in ref. [14]. We will use a slightly different formulation of this approach which is suitable to deal with different components of the moduli space.

Some aspects of Calabi-Yau fourfold compactifications of F-theory have been considered recently in refs. [15][16][17][18][19][20][21][22][23].

The plan of this paper is as follows. In section 2 we review how one can describe a six-dimensional compactification using the adiabatic arguments of [24]. One can fiber eight-dimensional theory data (it could be either F-theory or its heterotic dual) over a $\mathbf{P}^1$. F-theory is defined on elliptic Calabi-Yau threefold which is also a K3 fibration over $\mathbf{P}^1$. The base of the elliptic fibration is a rational ruled surface $\mathbf{F}_n$. The heterotic dual is characterized by the distribution of instanton numbers (12+n, 12-n) between two E_8's. Using the adiabatic arguments we give a nice geometric description of vector bundles on elliptic K3. This description is nothing else but the spectral cover theory [14] for $K3$. It allows us to reformulate various statements about F-theory – heterotic duality in a way preparing the reader for the more complex story awaiting him or her in four dimensions.

In section 3 we push the adiabatic argument further, down to four dimensions. In doing this we compactify heterotic string on the elliptic Calabi-Yau threefold. The F-theory dual is defined on Calabi-Yau fourfold which at the same time is a K3 fibration. The base of this elliptic CY fourfold is a $\mathbf{P}^1$ bundle over $\mathbf{F}_n$ which we call a generalized Hirzebruch variety $\mathbf{F}_{nmk}$, where the indices m, k indicate how the sphere $\mathbf{P}^1$ is fibered over $\mathbf{F}_n$. We discuss the spectral theory of vector bundles on elliptic Calabi-Yau threefolds and use it to describe various branches of their moduli spaces.

In section 4 we discuss various branches of the moduli spaces of the F-theory compactifications and relations between them. The full moduli space is a huge web which includes the moduli of the Calabi-Yau fourfold, gauge fields inside 7-branes and positions of 3-branes. All these moduli spaces are interrelated and the transition points correspond to the singularities in the moduli space. In the same context we discuss the relation between the 3-branes and the heterotic 5-branes. We present a map identifying the moduli

spaces of F-theory and heterotic compactification.

Section 5 deals with the moduli appearing in the transition from one branch to another. We compute the number of such moduli and explain their meaning both in F-theory and in heterotic string theory. We also address the general question about the matter content of F-theory.

In the Appendix we present various mathematical statements used in this paper. Detailed proofs of some of the theorems will appear elsewhere.

2. Review of F-theory – heterotic duality in six dimensions

2.1. Heterotic string

Here we briefly review F-theory – heterotic string duality in 6 dimensions but from a slightly different angle. This point of view allows us to generalize various six-dimensional results to four dimensions.

We first consider heterotic theory compactified on two-dimensional torus T^2 [25]. The heterotic string theory in 8 dimensions is uniquely defined by specifying a complex and Kähler structure on T^2 and a holomorphic $E_8 \times E_8$ bundle on T^2. The moduli space of $E_8 \times E_8$ bundles on T^2 is the same as the moduli of representations of $\pi_1(T^2)$ into $E_8 \times E_8$. The later is easily identified with $(\mathrm{Hom}(\pi_1(T^2), H))^W$ where $H \subset E_8 \times E_8$ is a Cartan torus and W is the Weyl group. Alternatively this moduli space can be written as

$$(\mathbf{C} \otimes \Gamma_{E_8 \times E_8})^W / H_1(T^2, \mathbf{Z}) \tag{2.1}$$

where $\Gamma_{E_8 \times E_8}$ is the co-root lattice of H and we have realized T^2 as the quotient $\mathbf{C}/H_1(T^2, \mathbf{Z})$.

Let us vary the eight-dimensional heterotic data over an additional $\mathbf{P}^1$ so that the family of complex tori fits into an elliptic $K3$. In order to formulate a heterotic string theory on this $K3$, we will also require that the $E_8 \times E_8$ bundles on the fibers fit into a global holomorphic bundle, which we denote by $\mathcal{V} = V_1 \times V_2$.

It would be desirable to have a description of $\mathcal{V}$ in terms of information concentrated along the fibers and the base of the $K3$ surface. By restricting $\mathcal{V}$ on the fibers and on the zero section of $K3 \to \mathbf{P}^1$ we obtain a family of flat bundles on the fibers and a flat bundle on the base[2]. Naively one would expect that this collection of data suffices to reconstruct $\mathcal{V}$. However, the information captured by these restrictions is incomplete and does not reflect the monodromy of the connection matrices on $\mathcal{V}\mid_{T^2}$ when we go around some special points on the base.

[2] By a flat bundle on a Riemann surface we mean a principal bundle admitting a holomorphic flat connection. In particular, the bundles $\mathcal{V}\mid_{T^2}$ can be flat without the restriction of the instanton connection on $\mathcal{V}$ being flat.

The precise relation between the family $\{\mathcal{V}\mid_{T^2}\}$ of flat bundles and $\mathcal{V}$ can be made explicit. To simplify the exposition we will discuss only the $SU(r)$ and $SO(n)$ vector bundles. We start with the $SU(r)$ bundles. The moduli space of $SU(r)$ flat bundles over the torus T^2 is a complex projective space $\mathbf{P}^{r-1}$. One can think about a point in $\mathbf{P}^{r-1}$ as a set of r points $(x_1,\ldots,x_r)$ in the dual torus $\check{T}^2$, subject to a constraint $\sum x_i = 0$. The set $(x_1,\ldots,x_r)$ parameterizes the $SU(r)$ Wilson line around T^2.

When we have a family $\{\mathcal{V}\mid_{T^2}\}$ of flat $SU(r)$ bundles parameterized by the projective line $\mathbf{P}^1$, the above construction produces a *spectral curve* $\Sigma \subset \check{K}3$. *The dual* $K3$ denoted by $\check{K}3$ is the elliptic fibration over $\mathbf{P}^1$ obtained from the original elliptic $K3$ by replacing each fiber $T^2 \to \check{T}^2$. The spectral curve Σ projects onto $\mathbf{P}^1$ so that the preimage of a point $p \in \mathbf{P}^1$ is the set $(x_1,\ldots,x_r)$ corresponding to the restriction $\mathcal{V}\mid_{T^2}$ of $\mathcal{V}$ to the fiber T^2 over p. In general the spectral curve consists of several irreducible components Σ_i with multiplicities r_i. Multiplicity $r_i > 1$ means that for any $p \in \mathbf{P}^1$, the line bundle $x_i \in \check{T}^2$ can be found r_i times in the decomposition of $\mathcal{V}\mid_{T^2}$. Each curve Σ_i may cover the base several times n_i. The class of the spectral curve Σ is given by

$$\Sigma = rS + kF \ , \tag{2.2}$$

where S is the zero section and F is the fiber and $r = \sum r_i n_i$. The coefficient k should be identified with $c_2(V)$.

The concept of spectral curve is very useful when we compare the heterotic compactification on $K3$ with the dual F-theory compactification on a Calabi-Yau threefold CY_3. What the adiabatic argument [24] essentially tells us is that the spectral curve Σ (together with $K3$) *determines the complex structure of the F-theory Calabi-Yau threefold.* To be precise, the complex moduli space of elliptic Calabi-Yau threefolds $\mathcal{M}^{(3)}$ can be represented as a bundle $(\mathcal{M}^{(3)} \to \mathcal{M}_{K3})$, where the fiber $\mathcal{M}_\Sigma$ is the moduli space of the spectral surface Σ and $\mathcal{M}_{K3}$ is the moduli space of complex structures of $K3$. The number of complex deformations of the singular locus of Calabi-Yau threefold coincides with the number of complex deformations of the spectral curve, which is equal to its arithmetic genus (see Appendix)

$$g(\Sigma) = \frac{1}{2}\Sigma^2 + 1 = rk - r^2 + 1 \ . \tag{2.3}$$

In the generic situation the spectral curve Σ consists of two curves with multiplicities one, each corresponding to the bundles $V_{1,2}$.

The information encoded in the spectral curve $\Sigma(\mathcal{V})$ is not sufficient to reconstruct the bundle $\mathcal{V}$ on $K3$. The moduli space of vector bundles on K3 is a hyperkähler variety and $\mathcal{M}_\Sigma$ is not hyperkähler (this variety is a projective space). Also the complex dimension of the moduli space of vector bundles is twice the dimension of $\mathcal{M}_\Sigma$. In fact, we have already encountered this situation in [26] in discussing the supersymmetric cycles in $K3$; so here we may simply borrow the result. A reader will find a more mathematically rigorous approach in the Appendix. It turns out that to recover the full moduli space the spectral curve Σ should be equipped with a line bundle L of degree $\deg L = -(r+g-1)$. The pushforward

of this line bundle on the base yields a vector bundle of rank r which coincides with the restriction of $\mathcal{V}_S$ to the zero section.

Proposition 1 stated in the Appendix reads that a pair (Σ, L) uniquely determines the vector bundle V with the trivial first Chern class and the second Chern class equal to $c_2(V) = k$. It is quite remarkable that the genus of the spectral curve is equal to half of the dimension of the moduli space of the vector bundle.

The number of matter multiplets can also be computed in terms of spectral curves. Suppose that the bundle V is a G-bundle where G is the broken subgroup of E_8. Let H be the unbroken subgroup so that $H \times G \subset E_8$. Let S_i be the representations of G, entering into the decomposition $\mathbf{248} = \bigoplus (S_i \otimes R_i)$, where R_i are the representations of the unbroken group H. With each representation $S_i(V)$ one can associate a spectral curve Σ_i. It follows from the index theorem [14] that the number of matter multiplets in representation R_i of H equals

$$N(R_i) = \Sigma_i \cdot S \ . \tag{2.4}$$

It is also quite interesting to consider the case of $SO(n)$ bundles[3]. One can describe $SO(n)$ flat bundles on a torus in terms of n points, invariant under the $\mathbf{Z}_2$ involution. The involution flips the sign of the flat coordinate along the torus ($\mathbf{Z}_2 : z \to -z$). This description gives rise to a $\mathbf{Z}_2$ invariant spectral curve Σ which covers the base n times. $\mathbf{Z}_2$ involution permutes the sheets when n is even. In the case when n is odd the spectral curve is reducible $\Sigma = S + \Sigma_{n-1}$ with the zero section S being fixed by the involution. The spectral curve Σ should be equipped with *anti-invariant* line bundle. The class of the spectral curve is equal to

$$\Sigma = nS + 2c_2(V)F. \tag{2.5}$$

Let us count the number of relevant deformations of the spectral curve Σ. Let $H^0(N_\Sigma)$ be the space of global sections of the normal bundle to Σ. Since the involution preserves Σ this space can be decomposed into the sum $H^0(N_\Sigma) = H^0_+(N_\Sigma) \oplus H^0_-(N_\Sigma)$, where $H^0_\pm(N_\Sigma)$ are the invariant (anti-invariant subspaces). The deformations we are interested in are the ones preserving the action of the involution and thus their number is equal to $H^0_+(N_\Sigma)$ for an even n and to $H^0_+(N_{\Sigma_{n-1}}(S))$ for an odd n. Alternatively in terms of the line bundle $\mathcal{O}(nS + 2c_2(V)F)$ with its natural $\mathbf{Z}_2$ action, the number of relevant deformations of Σ is given by $\dim H^0_-(K3, \mathcal{O}(nS + 2c_2(V)F))$ (regardless of the parity of n). If we consider the natural projection[4] $K3 \to \mathbf{F}_4 = K3/\mathbf{Z}_2$, then $nS + 2c_2(V)F$ is the preimage of the $\mathbf{Q}$-class $\sigma = (n/2)\underline{S} + 2c_2(V)\underline{F}$, where $\underline{S}$ and $\underline{F}$ are the infinity section and the fiber of $\mathbf{F}_4$, respectively. Now the dimension of $H^0_-(K3, \mathcal{O}(nS + 2c_2(V)F))$ can be easily computed

[3] These $SO(n)$ bundles are associated with $Spin(n)$ bundles, imbedded into E_8.

[4] The quotient of $K3$ by $\mathbf{Z}_2$ is a Hirzebruch surface $\mathbf{F}_n$. The zero section S of $K3$ is mapped to the infinity section $\underline{S}$ of $\mathbf{F}_n$. Computing the self-intersections we get $-2 = S^2 = (1/2)\underline{S}^2 = -n/2$. Therefore $n = 4$.

from the Lefschetz fixed point formula and is equal to

$$\dim H^0_-(K3, \mathcal{O}(nS + 2c_2(V)F)) = \frac{1}{2}(\sigma^2 - c_1\sigma) = c_2(V)(n-2) - n(n-1)/2 \ . \qquad (2.6)$$

Again, it is quite remarkable that $\dim H^0_-(K3, \mathcal{O}(nS + 2c_2(V)F))$ coincides with half of the dimension of the $SO(n)$ instanton moduli space. The dimension of the moduli space of anti-invariant line bundles also coincides with (2.6), which is just a consequence of the fact that the full moduli space is hyperkähler.

It is clear from the construction that the spectral curve Σ (collection of curves $\{\Sigma_i\}$ with multiplicities r_i) encodes the information about the Wilson lines along the fiber. Consider the case when several components of the spectral curve Σ, say Σ' and Σ'' degenerate to the one component with nontrivial multiplicity (for example $\Sigma' + \Sigma'' \to 2\Sigma$). As a result of this degeneration two line bundles L' and L'' are combined into a rank 2 bundle. In general, the degeneration of several components of the spectral curve Σ_i to a multiple one of the form $n\Sigma$ yields a more complicated object, e.g. a rank n vector bundle on Σ. In all these cases various Wilson lines get aligned and one should expect gauge symmetry enhancement. In fact, this mechanism looks very similar to the one responsible for the appearance of enhanced gauge symmetry when several parallel D-branes come together [9]. Comparing with F-theory side we can identify each degeneration of the spectral curve as some degeneration of the discriminant locus when several D-branes come together. We will discuss this phenomenon in detail in the context of four-dimensional F-theory compactifications.

2.2. F-theory

F-theory in 8 dimensions is defined on an elliptic $K3$. The moduli space of elliptic $K3$ surfaces is known to be the quotient of the symmetric space $SO(2,18)/SO(2) \times SO(18)$ by a discrete group (see [27][28] and references therein). It is a little bit better to think of this space as the bounded symmetric domain

$$\mathcal{D} = \{w \in \mathbf{P}((\Gamma_{E_8 \times E_8} \oplus \sigma \oplus \sigma) \otimes \mathbf{C}) \mid \langle w, w \rangle = 0, \langle w, \bar{w} \rangle > 0\}. \qquad (2.7)$$

The universal cover $(\Gamma_{E_8 \times E_8} \otimes \mathbf{C}) \times SO(2,2)/(SO(2) \times SO(2))$ of the moduli space of the heterotic string in 8 dimensions can be identified with $\mathcal{D}$. Therefore every variation of eight-dimensional heterotic data will produce a family of elliptic $K3$ surfaces. If the variation leads to a six-dimensional heterotic theory then the corresponding family of elliptic $K3$'s should fit into a Calabi-Yau threefold CY_3. Compactifying F-theory on this Calabi-Yau threefold, one gets F-theory in 6 dimensions, which is dual to the heterotic theory on $K3$. The complex structure of Calabi-Yau threefold is determined by the family of the heterotic data on the fiber T^2 which varies over $\mathbf{P}^1$. Therefore only a part of the full information about the vector bundle $\mathcal{V}$ on $K3$ encodes the threefold CY_3. This partial information is a family of Wilson lines $\{\mathcal{V}\mid_{T^2}\}$ for every fiber or equivalently the spectral curve. To sum

up, the elliptic fibration on the F-theory side is determined completely by the spectral curve Σ together with heterotic $K3$.

Now let us discuss the rôle of the line bundle L on Σ, which is necessary to reconstruct $\mathcal{V}$. The parameters in the polynomials in Weierstrass form correspond to complex scalar fields of the resulting $N = 1$ six-dimensional theory, each making half of hypermultiplet [1]. The other half of the hypermultiplets seem to be missing. To recover the missing complex parameters, one has to take into account that each 7-brane is equipped with $U(1)$ gauge field. Upon compactification down to six dimensions, exactly two components of the vector field become scalars and can be rearranged into one complex scalar field completing the hypermultiplet. As we mentioned before, the spectral curve Σ encodes the discriminant locus (locations of 7-branes). The bundle L on the spectral curve Σ completes the data so that the full moduli space is hyperkähler. Therefore it is clear that the line bundle L encodes information about the gauge bundle inside the 7-brane. To be more precise, the moduli space of the gauge fields inside 7-brane coincides with the moduli space of bundles on the spectral curve Σ.

Consider the simple example of heterotic theory with $\mathcal{V} = SU(n') \times SU(n'') \subset E_8 \times E_8$ bundle. Suppose that the unbroken gauge group is $G' \times G''$. The spectral curve Σ consists of at least two curves Σ', Σ'', each covering the base n' and n'' times. In the F-theory the discriminant locus in general consists of three 7-branes: D' and D'' with G' and G'' singularities and D_0 with generic I_1 singularity. It is easy to check that the number of deformations preserving $G' \times G''$ locus matches exactly the number of deformations of the curves Σ' and Σ'' inside $K3$. We will discuss similar counting in full generality in the case of four-dimensional compactifications. The $U(1)$ gauge bundle inside D_0 is determined by the line bundles on Σ' and Σ''.

3. Examples of F-theory – heterotic duality in four dimensions

3.1. Vector bundles and heterotic compactification

To describe the four-dimensional compactifications we fiber eight-dimensional data over a two-dimensional complex base $B_H = \mathbf{F}_n$. Suppose that eight-dimensional data fits into an elliptic Calabi-Yau threefold CY_3 and a vector bundle $\mathcal{V} = V_1 \oplus V_2$.

The description of vector bundles on the elliptic Calabi-Yau threefolds is more involved than the analogous construction for $K3$ (one can find a mathematical discussion in the Appendix). The discrete invariants of a vector bundle V with the trivial first Chern class are the rank r of the bundle, $c_3(V)$ and the components of $c_2(V)$. For the threefolds that we consider, this produces five integer parameters. The vector bundle is determined by the spectral surface Σ in the dual Calabi-Yau $\check{CY}_3$ and a line bundle $\mathcal{L}$ on a smooth model Y of the fibered product $\Sigma \times_{B_H} CY_3$. More precisely the bundle V is the pushforward of $\mathcal{L}$ from Y to CY_3. For every point $p \in CY_3$ the fiber of V is given by $x_1 + x_2 + \ldots + x_r$, where x_i are the fibers of $\mathcal{L}$ over the points $p_i \in \Sigma$, such that all p_i and p project on the same

point on the base B_H. Let us denote by L the restriction of the bundle $\mathcal{L}$ to the spectral surface Σ. In the case of elliptic K3 the bundle $\mathcal{L}$ can be uniquely recovered from L by pulling back and twisting with a certain fixed line bundle (see Appendix). In contrast, in the case of Calabi-Yau threefold there is no unique way to reconstruct $\mathcal{L}$ from the pullback of L, because one can also twist by the multiples of the exceptional divisor [5]. This twisting governs the third Chern class of the bundle V and has no effect on $c_1(V)$ and $c_2(V)$. It follows from the construction (see Appendix) that all deformations of the bundle $\mathcal{L}$ come from the deformation of the line bundle $L \to \Sigma$. Therefore, to simplify the discussion we can pretend that the vector bundle is determined by the line bundle L, keeping in mind that this is literally true only if there is a relation between $c_3(V)$ and $c_2(V)$.

The homological class of the spectral surface is determined by the rank of the bundle and its second Chern class

$$\Sigma = rS + c_2(V)_{AS}A + c_2(V)_{BS}B \ , \tag{3.1}$$

where r is the rank of the bundle and $c_2(V)_{AS,BS}$ are the coefficients in the decomposition of c_2 with respect to a basis AB, AS and BS. The cycles A, B and S make the basis in $H_2(CY_3)$ (see Appendix). The simplest way to derive (3.1) (in the case $n = 0$) is to restrict the bundle to the cycles A and B. These cycles can be represented by $K3$ and therefore we can use the relation (2.5). In general, for $n \neq 0$ the cycle $B + \frac{n}{2}A$ can be represented by $K3$ and the coefficients in (3.1) are determined by restricting the bundle V to the cycles A and $B + \frac{n}{2}A$.

The spectral surface Σ encodes *the complex structure* of the Calabi-Yau fourfold CY_4 in F-theory. Let us denote by $\mathcal{M}_\Sigma$ the moduli space of complex deformation of the spectral surface. Then the moduli space $\mathcal{M}^{(4)}$ of complex structures of the Calabi-Yau fourfold can be described as the bundle $(\mathcal{M}^{(4)} \to \mathcal{M}^{(3)})$ with a fiber $\mathcal{M}_\Sigma$, where $\mathcal{M}^{(3)}$ is the moduli space of complex structures of Calabi-Yau threefold CY_3 of the heterotic compactification.

When V is an $SU(r)$ vector bundle, the number of complex deformations of the spectral surface follows from the Riemann-Roch theorem and is equal to

$$\dim H^{(2,0)}(\Sigma) = \frac{1}{12}(2\Sigma^3 + \Sigma \cdot c_2(T_{CY})) - 1 \ . \tag{3.2}$$

In the next section we show that $\dim H^{(2,0)}(\Sigma)$ matches exactly the number of complex deformations preserving the gauge symmetry enhancement locus on the F-theory side. Expression (3.2) computes the number of deformations under the assumption that the spectral surface Σ is generic and irreducible. As we will see in the next section when the spectral surface degenerates and becomes reducible one has to compute the number of deformations *preserving* the number of irreducible components.

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[5] We do not see this ambiguity in the K3 case because the fibered product $\Sigma \times_{B_H} K3$ is already smooth.

Let us briefly describe what is going on in the case of $SO(n)$ bundles. As in six dimensions the spectral surface should be invariant under $\mathbf{Z}_2$ involution. The spectral surface should also be equipped with *anti-invariant* line bundle. It turns out that not every $\mathbf{Z}_2$ invariant spectral surface allows an *anti-invariant* line bundle. To count the number of parameters of spectral surfaces admitting an anti-invariant line bundle one needs to solve an explicit Noether-Lefschetz problem. Therefore this situation differs from the six-dimensional one. We will discuss the details of $SO(n)$ computations in [29].

The bundle L on Σ encodes the information about *the gauge fields* inside 7-branes. It is convenient to break the description into different sectors according to the different geometric behavior of the components of the spectral surface. For simplicity we will discuss only the two limit cases. First, suppose that the surface Σ' is a component of the spectral surface Σ with multiplicity one. Then the line bundle $L' = L_{|\Sigma'}$ has no deformations because $h^1(\Sigma') = 0$. This corresponds to the fact that the $U(1)$ gauge field inside the 7-brane has no extra moduli. The situation is different when the multiplicity of Σ' is $n > 1$. In this case the non-reduced surface $n\Sigma'$ is equipped with a sheaf L' that has numerical rank one (as measured by the Hilbert polynomial). The space of such sheaves has several connected components labeled by the collections of ranks and degrees of the restrictions of L' on the infinitesimal neighborhoods of $\Sigma' \subset \check{CY}_3$ of orders $0, 1, 2, \ldots, n-1$. For example we will have a component parameterizing rank n vector bundles on the reduced surface Σ' and a component parameterizing all line bundles on the full non-reduced surface $n\Sigma'$. In the case of a 6-dimensional compactification, when $n\Sigma'$ is multiple curve sitting on a (not necessarily elliptically fibered) $K3$ surface a detailed description of these components in their structure can be found in [30]. It is very interesting to find the branches of the F-theory compactifications corresponding to these components. The analysis of the geometry of these branches is rather involved and will be a subject to a future investigation [29]. For now we will examine the special case when the sheaf L' is determined by a rank n vector bundle M on Σ', which is characterized by $c_2(M)$. To explain how this vector bundle appears let us consider a concrete example. Let $V = \pi^* M$ be a vector bundle on the Calabi-Yau threefold CY_3 which is a pullback of the bundle M on the base B_H; we denote by π the projection $\pi : CY_3 \to B_H$. Restricted to any fiber, $\pi^* M \mid_{T^2}$ is a trivial rank n bundle. The corresponding spectral surface is the zero section S taken with multiplicity n: $\Sigma(\pi^* M) = nS$. The spectral bundle on S is M itself, considered as a bundle on the zero section.

To appreciate the rôle of bundle M on the spectral surface with multiplicity, let us consider another example. Choose a special vector bundle $V = E \oplus \cdots \oplus E = E \otimes I_n$ where E is any irreducible bundle and I_n is a trivial vector bundle of rank n. The spectral surface of V is the same as the spectral surface of E taken with multiplicity n. The bundle V has a large group of automorphisms which acts on I_n. In this example the group of automorphisms is $SU(n)$. In heterotic compactification when V is used to gauge $E_8 \times E_8$, the automorphism group is part of an unbroken gauge symmetry. This implies that in the dual F-theory there is a 7-brane which carries this particular gauge symmetry group.

Now consider the deformations of M *preserving* the multiple component Σ' of the spectral surface. Such deformations are described exactly by the moduli of the bundle M on Σ'. In general the deformations break the structure of the product $E \otimes I_n$ so the automorphism group of V, which is gauge symmetry in 4 dimensions, can become smaller or disappear altogether. This should be compared to the symmetry breaking mechanism by the instanton background inside the 7-brane.

This example shows that the bundle M on the spectral surface is in a one-to-one correspondence with the gauge bundle inside the 7-brane in F-theory. In particular the Chern classes of the two bundles are related. The precise map between the bundle on the spectral surface Σ' and the gauge fields inside 7-branes can be quite complicated. We are planning to return to this discussion in one of our future publications [29]. In the section 4 of this paper we will find this map in one simple but very important example describing the 3-(5-)brane-instanton transition.

We see that the full moduli space of vector bundles on the elliptic Calabi-Yau threefold is a huge web which contains various irreducible components. Moreover, this space has a natural stratification. Each stratum is characterized by the number of irreducible components of the spectral surface, their multiplicities and the second Chern classes $c_2(M_j)$ of the spectral bundles on multiple components. The strata are connected through the transition points. Some of these transitions are discussed below.

3.2. Singularities of elliptic fibration

F-theory is compactified on elliptic Calabi-Yau fourfold with a section. For practical reasons we represent the elliptic fibration in the Weierstrass form

$$y^2 = x^3 + xf(\cdot) + g(\cdot) \ , \tag{3.3}$$

where $f(\cdot)$ and $g(\cdot)$ are the polynomials on the base. The base of the elliptic fibration is a complex manifold, which has the structure of $\mathbf{P}^1$ bundle over $\mathbf{F}_n$. Let us denote the coordinate along the fiber as z, and the coordinate along the base as w, u. The polynomials $f(\cdot)$ and $g(\cdot)$ have the following expansions

$$f(z,w,u) = \sum_{a=1}^{8} z^a f_a(w,u) \ , \quad g(z,w,u) = \sum_{b=1}^{12} z^b g_b(w,u). \tag{3.4}$$

As suggested in [1], the polynomials with $a < 4$, $b < 6$ and $a > 4$,$b > 6$ encode the information about the bundles $V_{1,2}$. Polynomials $f_4(w,u)$ and $g_6(w,u)$ govern the complex moduli of Calabi-Yau threefold of the heterotic compactification.

The singularity of the elliptic fibration along a section $z = z(w,u)$ can result in the perturbative gauge symmetry observed in heterotic compactification. It depends on the gauge bundle inside the 7-brane whether the full symmetry is observed. If the bundle is trivial, the singularity structure governs the gauge group, otherwise the gauge group is broken by instantons.

In the above example the 7-brane covers the base space F_n. The other possibility for the singularity of the elliptic fibration is to occur along the divisor D that projects to a curve X on the base. Such singularity corresponds to a nonperturbative effect on the heterotic side. Namely, when the discriminant has zero of order greater than 1 along D, the heterotic string contains a 5-brane wrapped around the curve X [1][2].

The description of vector bundle on elliptic threefold in terms of the spectral surface Σ equipped with the bundle allows us to identify the degrees of freedom that correspond to the complex structures on the F-theory fourfold CY_4. Consider the situation when $E_8 \times E_8$ is broken by bundle $\mathcal{V} = V_1 \oplus V_2$ down to $G_1 \times G_2$. Similarly to the six-dimensional compactifications, the decomposition of the second Chern class $c_2(\mathcal{V})$ into $c_2(V_1)$ and $c_2(V_2)$ is fixed by the F-theory data:

$$c_2(V_{1,2}) = x_{1,2}AB + (12 + 6n \pm m)AS + (12 \pm k)BS \; . \tag{3.5}$$

In the case when $n = 0$, one can easily derive this decomposition by restricting the vector bundles $V_{1,2}$ on the cycles A and B both representing $K3$. Elliptic CY_3 in question is a *double* $K3$ fibration. The vector bundles $V_{1,2}$ can be described by fibering the restriction of the bundle on either of these $K3$s. Therefore the decomposition of the Chern classes along both $K3$s implies (3.5) (see [1]). This arguments can also be generalized for $n \neq 0$ (cf. the explanation after (3.1)). The only unfixed coefficient is the one in front of AB (projection along the elliptic fiber). The sum of these two coefficients is related to the number of 5-branes wrapped around the elliptic fiber in the compactification in question. Namely[6],

$$N(\text{branes}) = c_2(T_{CY_3})_{AB} - x_1 - x_2 = \frac{\chi(CY_4)}{24} - \tilde{c}_2 \; , \tag{3.6}$$

where $c_2(T_{CY_3})_{AB}$ is the coefficient in front of AB in the decomposition of $c_2(T_{CY_3})$ with respect to a basis AB, AS and BS. Expression (3.6) equates the number of 3-branes in F-theory compactification with the number of 5-branes in the heterotic compactification. In the case of singular Calabi-Yau threefold the Euler character can be computed using the methods discussed in [23]. The last term on the r.h.s. counts the number of instantons inside 7-branes.

Two bundles $V_{1,2}$ enter on the equal footing and therefore we may discuss just one of them. It follows from (3.1) that the class of the surface $\Sigma(V_1)$ is equal to

$$\Sigma(V_1) = r_1 S + (12 + k)B + (12 + 6n + m)A. \tag{3.7}$$

From the equation (3.7) it follows that $\Sigma(V_1)$ is a zero set of the polynomial (cf. eq. (2.6) in [14])

$$a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 x^2 y + \cdots, \tag{3.8}$$

[6] The second chern class of the tangent bundle of elliptic Calabi-Yau is equal to $c_2(T_{CY}) = 92AB + (24 + 12n)AS + 24BS$.

the last term is either $a_r x^{\frac{r}{2}}$ for even r or $a_r x^{\frac{r-3}{2}} y$ for odd r. The coefficients a_q are the sections of the line bundle $K_B^{q-6} \otimes \mathcal{O}(m\,a + k\,b)$ on the base F_n, the line bundle $K_B = \mathcal{O}(-2b - (n+2)a)$ is the canonical bundle on F_n. In other words, the coeffficients a_q are polynomials in $w,\ u$ given by

$$a_q(w,u) = \sum_i^{12-2q+k} w^i \sum_j^{12-4i+(6-2i)n+m} a_{qij} u^j. \tag{3.9}$$

The number of complex deformations of the spectral surface Σ is given by the adjunction formula (3.2). To illustrate our point let us compare the number of complex deformations in F-theory with the number of complex deformation of the spectral surface Σ on the heterotic side. We summarize the number of complex deformations *plus* 1 in the table below. These numbers are derived under the assumption that the spectral surface is irreducible.

Table 1.

	Unbroken subgroup	$\frac{1}{12}(2\Sigma^3 + \Sigma \cdot c_2(T_{CY}))$
$r=1$	E_8	$169 + 13(k+m) + km - nk(13+k)/2$
$r=2$	E_7	$250 + 22(k+m) + 2km - nk(11+k)$
$r=3$	E_6	$299 + 29(k+m) + 3km - nk(29+3k)/2$
$r=4$	$SO(10)$	$324 + 34(k+m) + 4km - nk(17+2k)$
$r=5$	$SU(5)$	$333 + 37(k+m) + 5km - nk(37+5k)/2$
$r=6$	$SU(2) \times SU(3)$	$334 + 38(k+m) + 6km - nk(19-3k)$
$r=7$	$SU(2) \times U(1)$	$335 + 37(k+m) + 7km - nk(37-7k)/2$

Let us compare these expressions with the similar computations on the F-theory side. Elliptic fibration is given in Weierstrass form (3.3), where

$$\begin{aligned} f(z,w,u) &= \sum_a z^a \sum_i^{8+k(4-a)} w^i \sum_j^{8+m(4-a)+n(4-i)} u^j f_{aij}\ , \\ g(z,w,u) &= \sum_b z^b \sum_i^{12+k(6-b)} w^i \sum_j^{12+m(6-b)+n(6-i)} u^j g_{bij}\ . \end{aligned} \tag{3.10}$$

Below we compute the complex deformations *preserving* a particular singular locus. In doing that we just compute the number of deformations of polynomials (number of coefficients) that does not affect the singularity structure. The number of coefficients differs from the number of complex deformations by 1, which is due to the possibility of rescaling

the z coordinate; this does not affect the position and the structure of the singularity. For this reason Table 1 gives the number of complex deformations *plus* 1.

Note that one can identify the Kodaira type of a singular fiber in an elliptic fibration by using Tate's algorithm [31]. In the examples considered in this section the conditions for having a *split* singularity turn out to be the same as in 6 dimensions [2]. This determines the unbroken gauge groups as given in Table 1.

We start with the most singular case when elliptic fibration has E_8 singularity located at zero section ($z = 0$). The singularity located at the section at infinity corresponds to the other bundle, say V_2 and is irrelevant for our considerations. The singularity is characterized by polynomials $f_a(w,u)$ and $g_b(w,u)$ with $a \geq 4,\ b \geq 5$. The number of complex deformations *preserving* E_8 locus is equal to the number of coefficients in $g_5(w,u)$ (the rest of the polynomials specifies other data) and it is given by

$$\sum_{i=0}^{12+k}(13+m+n(6-i)) = 169+13(k+m)+km-nk(13+k)/2\ , \tag{3.11}$$

provided that $12+m \geq 6n+kn$. Comparing this with (3.9) we see that one can identify $g_5(w,u)$ with $a_0(w,u)$ in (3.8).

As one can see from Table 1, the E_8 singularity in F-theory formally corresponds to the rank 1 heterotic bundle. To understand this special situation, let us return to the F-theory/heterotic duality in six dimensions. The E_8 singularity on the F-theory side is interpreted there in terms of the zero size E_8 instantons on the heterotic side [1]. The new physics could be described by tensionless strings. Let us see how the zero size instantons appear in the spectral theory of $K3$. The spectral curve in the dual $\check{K}3$ for $r=1$ is given by $\Sigma = S + pF,\ p = 12 \pm n$. It is reducible: one irreducible component is the zero section S and p other irreducible components are fibers $\check{T}^2_i,\ i = 1,\ldots,p$. The elliptic components carry line bundles $L_i \in T^2_i$ which can be identified with points on the fibers T^2_i of the physical $K3$. This spectral data corresponds not to a bundle, but to a torsionless sheaf with pointlike instantons (5-branes) located in the points L_i on the fibers T^2_i. Deformations of the spectral curve Σ move the 5-branes along the base $\mathbf{P}^1$. Deformations of the spectral bundle L_i move the i-th 5-brane along the fiber. The dimension of the moduli space $\mathcal{M}_\Sigma$ is p which coincides with the formal genus of Σ. The full moduli space is birational to the symmetric product $\mathrm{Sym}^p K3$ and has the dimension $2p$.

Now let us return to four dimensions. We will interpret the E_8 singularity of CY_4 in terms of the instanton (5-brane) wrapped around the curve $C = (12+6n+m)AS + (12+k)BS$ in the base $\mathbf{F}_n$. The spectral surface Σ of this bundle has two irreducible components. One is the zero section S which is rigid. The other is a preimage of the curve C. The deformations of C along the base F_n are described by the coefficients in the polynomial $g_5(z,w)$. Their number is given by (3.11).

For the heterotic bundle of rank 2, one expects to get the E_7 singularity in the F-theory compactification. According to Kodaira classification in the case of E_7 singularity

all terms $f_a(w,u)$ and $g_b(w,u)$ with $a<3$ and $b<5$ vanish. Again, we assume that n,m,k satisfies some relations, namely $8+m \geq 4n+kn$, $12+m \geq 6n+kn$. The deformations are described by the polynomials $g_5(w,u)$ and $f_3(w,u)$ which one can identify with $a_0(w,u)$ and $a_2(w,u)$ in (3.8). Thus the $SU(2)$ spectral surface corresponding to the E_7 singularity is given by

$$g_5(w,u)+f_3(w,u)x=0. \tag{3.12}$$

In this domain the number of parameters is given by

$$\sum_{i=0}^{8+k}(9+m+n(4-i))+\sum_{i=0}^{12+k}(13+m+n(6-i))$$

which, of course, is the number of deformations for $r=2$ (see Table 1).

For E_6 singularity there is an extra term with $g_4(w,u)$. This is the first case when the polynomial is not generic and one has to impose an extra condition (generic polynomial corresponds to F_4 singularity). The polynomial $g_4(w,u)$ should be a perfect square [2] and it could be written as $g_4(w,u)=q(z,w)^2$, where

$$q(w,u)=\sum_{i}^{6+k} w^i \sum_{j}^{6+m+n(3-i)} u^j q_{ij} \ . \tag{3.13}$$

The polynomial $q(w,u)$ can be identified with $a_3(w,u)$, so the $SU(3)$ spectral surface corresponding to the E_6 singularity is given by

$$g_5(w,u)+f_3(w,u)x+q(w,u)y=0. \tag{3.14}$$

This polynomial $q(w,u)$ produces $\Delta=49+7(k+m)+km-nk(7+k)/2$ extra parameters, provided that $6+m \geq 3n+kn$. One can easily see that Δ is exactly the difference between $r=3$ and $r=2$ (see Table 1.). It is remarkable that the conditions on the polynomials of the elliptic fibration found in [2] do not depend on the dimension of compactification.

Let us also check the rank $r=4$ bundle (corresponding to $SO(10)$ singularity). There are two additional terms to take into account $f_2(w,u)$ and $g_3(w,u)$. These terms are not independent but should be related $f_2(w,u)=h^2(w,u)$ and $g_3(w,u)=h^3(w,u)$, where $h(w,u)$ has the following expansion

$$h(w,u)=\sum_{i}^{4+k} w^i \sum_{j}^{4+m+n(2-i)} u^j h_{ij} \ . \tag{3.15}$$

Sure enough, we can identify $h(w,u)$ with $a_4(w,u)$, so the corresponding $SU(4)$ spectral surface is

$$g_5(w,u)+f_3(w,u)x+q(w,u)y+h(w,u)x^2=0. \tag{3.16}$$

The polynomial $h(w,u)$ gives $25+5(k+m)+km-nk(5+k)/2$ extra parameters. Again, this is consistent with the results in Table 1.

Using the results of [2] one can also verify that the number of deformations preserving the gauge symmetry enhancement locus matches with the number of deformations of vector bundles of rank $r=5,6,7$. These calculations are straightforward and we do not present them here.

On physical grounds, one expects that the numbers of complex deformations in Table 1 should be consistent with Higgsing $E_7 \to E_6 \to SO(10) \to SU(5) \to SU(2)\times SU(3) \to SU(2)\times U(1)$, similarly to the six-dimensional case [2]. However, it is easy to check that for consistency in four dimensions one has to assume the existence of (quite nontrivial) superpotentials in the low-energy effective theory, which by F-flatness conditions would decrease the dimensions of Higgs branches. It would be interesting to investigate such superpotentials both in heterotic theory and in F-theory.

In the computations discussed above we assumed that parameters n, m and k satisfy some conditions. If these conditions are not satisfied then the summation limits in (3.10) become different. For simplicity, consider the case of $n=2$. It turns out that the both conditions discussed above are equivalent to $m \geq 2k$. If instead $m < 2k$, then the expansions (3.10) read as follows

$$\begin{aligned} f(z,w,u) &= \sum_a z^a \sum_i^{8+2m-[ma/2]} w^i \sum_j^{16+m(4-a)-2i} u^j f_{aij} \\ g(z,w,u) &= \sum_b z^b \sum_i^{12+3m-[mb/2]} w^i \sum_j^{24+m(6-b)-2i} u^j g_{bij} \ , \end{aligned} \tag{3.17}$$

where $[x/2]$ denotes the integer part of $x/2$. It is clear that the number of complex deformations is going to be different from the one computed above under the assumption $m \geq 2k$. We suggest the following explanation of this phenomenon. When $m < 2k$ the spectral surface becomes reducible. The class of the spectral surface is given by (3.7), but now it has two components Σ' and Σ'':

$$\Sigma' = rS + (24+m)A + (12+m/2)B \ , \quad \Sigma'' = (k-m/2)B \ . \tag{3.18}$$

It turns out that the surface Σ'' is rigid and therefore all deformations come from Σ'. One can easily verify that the number of deformations encoded in polynomials (3.17) exactly matches the number of deformations of the surface Σ'. The example presented here is very simple. In general, the spectral surface may have several irreducible components. It would be interesting to investigate this further.

3.3. Singularity vs. gauge group

Below we consider an example in which the 4d gauge group differs from the one, prescribed by the singularity. This example was constructed by M. Bershadsky, S. Kachru, V.

Sadov and C. Vafa (unpublished). The F-theory is compactified on generalized Hirzebruch with [7] $m = 12 + 6n$ and $k = 12$. In this case the elliptic fibration has an E_8 singularity along the section at infinity. The E_8 component does not intersect other components of the discriminant locus. We will consider a situation when that E_8 is not broken by instantons and the gauge group in four dimensions is $SU(5) \times E_8$. The factor $SU(5)$ corresponds to the 7-brane *with E_7 singularity* wrapped around zero section. The point here is that this 7-brane carries a nontrivial rank 3 instanton bundle so that a would-be E_7 is broken to $SU(5)$.

To describe this theory in the heterotic language we note that for $m = 12 + 6n$ and $k = 12$ the decomposition of the second Chern classes is given by (3.5)

$$\begin{aligned} c_2(V_1) &= x_1 AB + (24 + 12n)AS + 24BS \\ c_2(V_2) &= x_2 AB \ , \end{aligned} \tag{3.19}$$

where $x_1 + x_2 = (92 - N_5)$, where N_5 is the number of 5-branes. Since we want an unbroken E_8 in four dimensions, we choose the bundle V_2 to be trivial, so $x_2 = 0$. Also we want a *perturbative* heterotic compactification with no 5-branes which forces the condition $x_1 = 92$. Finally, $c_2(V_1) = c_2(CY_3)$.

The bundle V_1 is characterized by assigning to it the same toric data as for Calabi-Yau threefold

$$n_{iJ} = \begin{pmatrix} 1 & 1 & n & 0 & 4+2n & 6+3n & 0 \\ 0 & 0 & 1 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{pmatrix} \ , \tag{3.20}$$

where the index $J \in (1, 2, 3)$. Let us also define $m_J = \sum_i n_{iJ}$. The vector bundle is defined by the cohomology of the sequence

$$0 \to \mathcal{O} \to \bigoplus_i \mathcal{O}(\sum_J n_{iJ} X_J) \to \mathcal{O}(\sum_J m_J X_J) \to 0 \tag{3.21}$$

The classes X_J represent the familiar basis A, B and S in $H^4(CY_3)$. The map from $\bigoplus_i \mathcal{O}(\sum_J n_{iJ} X_J)$ to $\mathcal{O}(\sum_J m_J X_J)$ is given by the polynomials $F_i(\cdot)$ of three-degree $(m_1 - n_{i1}, m_2 - n_{i2}, m_3 - n_{i3})$. It is easy to see that $ch(V) = 2 + ch(CY_3)$ so one can think about this bundle as a deformation of the tangent $SU(3)$ bundle into an $SU(5)$ bundle. The gauge group $E_8 \times E_8$ is broken down to $SU(5) \times E_8$.

It turns out that the spectral surface for this bundle consists of two components Σ' and Σ'', where

$$\Sigma' = 2S + (24 + 12n)A + 24B \tag{3.22}$$

and the second component Σ'' is a zero section S with multiplicity 3. To see that one restricts (3.21) to the elliptic fiber realized as a degree 6 hypersurface

$$W_6 = y^2 - x^3 - f\, xz^4 - g\, z^6 = 0$$

[7] In order to obtain a nonsingular heterotic threefold, one needs to choose $n = 0, 1, 2$.

in the weighted projective space $W\mathbf{P}^2_{1,2,3}$. The sequence (3.21) becomes

$$0 \to \mathcal{O} \to \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3) \to \mathcal{O}(6) \to 0,$$

where the second map is given by polynomials $(E_1, ...E_7)$ of degrees $(0,0,0,0,1,2,3)$ and the map into $\mathcal{O}(6)$ given by polynomials $(F_1, \ldots, F_7)$ of degrees $(6,6,6,6,5,4,3)$ such that $\sum E_i F_i = 0$ (mod W_6). It is easy to check that by making appropriate field redefinitions in the linear σ-model $(F_1, \ldots, F_7) \to (\tilde{F}_1, \ldots, \tilde{F}_7)$ one can make $\tilde{F}_1 = 0, \ldots, \tilde{F}_4 = 0$. That implies that the spectral surface reduces to two components one of which is the zero section with multiplicity 3.

It is important that deformations of the spectral surface are given only by the deformations of the first component Σ', since Σ'' is rigid. Therefore, at the level of parameter counting, this example coincides with the one of the rank 2 bundle in spite of the fact that the rank of the bundle is 5. The counting for rank 2 was done in section 3 where we found that on the F-theory side this situation corresponds to E_7 singularity of Calabi-Yau fourfold!

The component Σ'' of the spectral surface carries a rank 3 spectral bundle which should be related to the instanton bundle on the E_7 D-brane. This bundle breaks E_7 to $SU(5)$. The deformations of V_1 comes from 1) the deformations of Σ' and 2) the deformations of the spectral bundle on Σ''. Only the deformations of the first type can be counted using the Poincare polynomial technique.

4. Mixed moduli space of F-theory in four dimensions

Here we shall discuss the branches of the moduli space of F-theory compactified on a fourfold CY_4. The possibility for various branches occurs when CY_4 has a singularity due to degeneration of elliptic fiber along a component of the discriminant locus. The 7-brane corresponding to that component carries a non-abelian gauge group. The moduli we want to discuss describe instantons on that 7-brane.

The Calabi-Yau manifold CY_4 is an elliptic fibration over the base B_F. If CY_4 is *also* a $K3$ fibration over the base B_H, it is conjectured to be dual to heterotic compactification on CY_3 — Calabi-Yau elliptic fibration over B_H [32] [33]. We assume that elliptic and $K3$ fibration structures are compatible so that the threefold B_F is a $\mathbf{P}^1$ fibration over B_H.

F-theory on CY_4 develops anomaly given by $\alpha = \chi/24$ where χ is the Euler character of CY_4 [5]. To cancel the anomaly one can put α 3-branes inside B_F. In the heterotic theory this means putting inside CY_3 α 5-branes wrapped around elliptic fiber. One expects a one-to-one correspondence between 3- and 5-branes in two theories. Therefore it is very instructive to compare the moduli space of 3-branes inside B_F with the moduli of 5-branes inside CY_3.

Let us start with smooth CY_4 and cancel the anomaly by 3-branes. In the heterotic theory $E_8 \times E_8$ is completely broken by the bundle $\mathcal{V}$. Choose a complex structure on CY_4.

Then the moduli space of a 3-brane is 3-dimensional: it is B_F — the base of the elliptic fibration CY_4. To see this moduli space in the 5-brane picture we recall that B_F is a $\mathbf{P}^1$ bundle over B_H. The *position* of a 5-brane is specified by a point on a 2-dimensional B_H. The coordinate of a 3-brane along the fiber $\mathbf{P}^1$ should be identified[8] with the position on the Coulomb branch of the moduli space of 5-brane compactified on T^2.

3-branes in F-theory are probes measuring the local geometry of elliptic fibration. Similarly, 5-branes are the probes in heterotic compactification, measuring geometry and the restriction $\mathcal{V}\mid_{T^2}$ of the heterotic bundle V to elliptic fiber. Compactified on a given fiber T^2 with given Wilson lines $\mathcal{V}\mid_{T^2}$, 5-brane has a 1-dimensional Coulomb moduli space parameterized by the superpartner of photon.

It should be noted that the effective 4-dimensional theory on the 3-brane probe has $N=2$ supersymmetry for the 3-brane located close to the 7-brane. The supersymmetry can be broken (by the background) to $N=1$ when a 3-brane approaches special divisors on 7-branes [34]. In the present discussion we restrict ourselves to generic situation.

Now we have a setup to describe other branches of the moduli space of F theory on CY_4. On these branches, the 3-brane anomaly is cancelled by both 3-branes and instantons inside 7-branes. To study the transition from the no-instantons branch to a branch with instantons let us consider a 3-brane probe in the vicinity of the 7-brane carrying a non-abelian gauge group G. The effective theory possesses the product gauge group $U(1)\times G$. Open strings connecting 3-brane with 7-brane produce a matter hypermultiplet charged with respect to both $U(1)$ and G, with mass proportional to the distance between these

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[8] There is a good understanding of such identification for the $SO(32)$ heterotic string 5-brane which carries a $SU(2)$ vector multiplet and a half-multiplet in $(\mathbf{2},\mathbf{32})$. The projective line $\mathbf{P}^1$ can be identified with the moduli space of the background $SU(2)$ bundle on T^2. In four points on $\mathbf{P}^1$, corresponding to four spin structures on T^2, the $SU(2)$ symmetry is restored *classically.* In quantum theory these points correspond to pure $N=2$ $SU(2)$ gauge theory. So on the *quantum* $\mathbf{P}^1$ each spin structure gives rise to two points separated by the mass gap Λ_i, $i=1,\ldots,4$. Also on $\mathbf{P}^1$ there are 16 points where the Wilson line of $SU(2)$ restores a $U(1)$ subgroup of $SO(32)$. A massless charged field which is a part of $(\mathbf{2},\mathbf{32})$ appears at these points. Altogether, there are $16+4\cdot 2=24$ special points on $\mathbf{P}^1$, as expected from 3-5-brane correspondence. The mass gaps Λ_i are not all independent, because three nontrivial spin structures of the fiber are permuted by monodromy around the discriminant locus. In fact, they fit into a surface which covers the base B_H three times. So only one of three $\Lambda_2,\Lambda_3,\Lambda_4$ is an independent parameter. Together with Λ_1 and 16 Wilson lines this gives 18 independent parameters, also as expected. Tuning 16 Wilson lines of $SO(32)$ one can find various theories with extended global symmetries. For example, consider a $SO(32)$ bundle on T^2 given by $I_8\otimes(L_1\oplus L_2\oplus L_3\oplus L_4)$ where I_8 is a trivial $SO(8)$ bundle and L_i are the four spin structures. The quantum moduli space is $\mathbf{P}^1$ with four special points corresponding to the spin structures, where the global $SO(8)\subset SO(32)$ is restored. At these points, the effective theory is the (finite) $N=2$ $SU(2)$ with four flavors.

D-branes. When the 3-brane probe is away from the 7-brane, the matter fields are heavy so that the $N = 2$ supersymmetric $U(1)$ theory on the probe is on the Coulomb branch. As the 3-brane approaches the 7-brane, the $G-$multiplet of $U(1)$ hypermultiplets becomes light and a transition to the Higgs branch is possible. Note that the $U(1)$ Higgs branch intersects the Coulomb branch in points where the number of massless hypermultiplets (the dimension of the G-multiplet) is at least two. Therefore, the group G has to be at least $SU(2)$.

To make the G-multiplet of matter fields massless, the 3-brane should sit on top of the 7-brane. It is known that such configuration of D-branes can be identified with a point-like G-instanton. Turning on the *vev* of the matter hypermultiplets means smoothing out the singular gauge fields corresponding to the point-like instanton [11][12]. Therefore, the Higgs branch of the effective theory consists of instantons of finite size of the nonabelian gauge group G on the 7-brane. At the transition point the $U(1)$ gauge group decouples (at least for $G = SU(n)$) and the Higgs branch is in fact the Higgs branch of the gauge theory with the gauge group G.

To sum up, the transition amounts to eating up k 3-branes and turning on k instantons inside the 7-brane with nonabelian gauge group G. The F-theory anomaly remains cancelled.

Consider an important example when the compact part of the 7-brane worldvolume is the zero section of the bundle $B_F \to B_H$, so it can be identified with B_H itself. Suppose that the singularity of the elliptic fibration along this 7-brane is such that in the absence of instantons there is a gauge group G in four dimensions. Let us take k 3-branes on top of this 7-brane, so that the "3-brane group" is $SU(k)$. Each 3-brane produces a hypermultiplet (a pair of chiral fields). These hypermultiplets are massless states of the open strings connecting 3-branes and 7-branes. The "3-brane end" of the string carries the flavor index of $SU(k)$, while the "7-brane end" is charged with respect to the gauge group on the 7-brane. Giving *vev* to the hypermultiplets, one makes the 3-brane-instanton transition. In the 4-dimensional field theory language, the gauge group gets broken by the nonzero *vev*'s. In the D-brane language, it is broken by the G-instantons on the 7-brane.

These two descriptions should match. For $G = SU(n)$ the dimension of the instanton moduli space is equal to $2nk - (n^2 - 1)\,(1 - h^{01} + h^{02})$. Taking into account that for B_H the Hodge numbers $h^{01} = h^{02} = 0$, we arrive at

$$\dim\mathcal{M} = 2nk - (n^2 - 1)\ . \tag{4.1}$$

This formula has indeed a very clear interpretation in terms of Higgs mechanism [21]: it counts the dimension of the Higgs branch of $SU(n)$ gauge theory[9]. The instanton number k coincides with the number of 3-branes and therefore counts flavors: $N_f = k$ since

[9] For a recent discussion of SO and Sp cases see ref. [22]. A consideration of exceptional groups would involve tensionless strings [1].

each 3-brane produces two chiral fields (N_f counts the number of pairs of fundamental-antifundamental representations).

When $N_f < n$ there are no flat directions due to the nonperturbative superpotential $\sim 1/(Q\tilde{Q})^{1/(n-N_f)}$ where Q, $\tilde{Q}$ are squark chiral superfields. The appearance of such superpotentials is well known in field theory [35]. In the case of massless squarks this superpotential lifts the ground state of the field theory. When squarks have finite non-zero masses the theory has a stable vacuum corresponding to nonzero expectation values of squarks. In the context of F-theory for $N_f = n - 1$, this superpotential was recently discussed in [21]. The bare masses of squarks are proportional to distances between 7- and 3-branes. Therefore at the level of effective field theory the limit of vanishing distance between 7- and 3-branes is ill-defined and may require various quantum corrections that stabilize[10] the vacuum of F-theory at $< Q\tilde{Q} >\sim M_s^2$, where M_s is the string scale. In the context of $(0,2)$ heterotic compactifications, this question was recently discussed in ref. [36].

In general, apart from the non-chiral matter coming from 3-branes (we call this matter Type A), there are chiral matter fields (we call them Type B) which come from the intersections of the given 7-brane with other 7-branes[11]. In the effective $N = 1$ supersymmetric 4-dimensional theory there is a tree level superpotential $W = W(A, B)$ which couples these two sorts of matter. This superpotential implies that when the Type A fields develop nonzero *vev*'s, some of the Type B fields may become massive. The origin of this superpotential can be more clearly seen in the D-brane language. Nonzero *vev*'s of the Type A fields correspond to the nontrivial instanton background in the compact directions. Type B multiplets are charged with respect to the gauge group so that they interact with the instanton field and become massive.

The nonzero *vev*'s of Type B fields also break the gauge symmetry. In the D-brane language these *vev*'s generate splitting of the 7-brane with charge $Q_{(7)} > 1$ to several 7-branes with smaller charges. Concretely, Type B matter is a source term in the generalized Hitchin equations describing fields in the bulk of the 7-brane. We call fields in the bulk Type C matter. Nontrivial solutions for these Type C fields which transform in the adjoint of the gauge group correspond to the splitting of the 7-brane. We will give a more detailed exposition on this in the section 5.

As an immediate consequence of the above discussion we see that when the *vev*'s of the Type A fields are turned on (=there is nontrivial instanton background), the moduli of complex deformations of CY_4 which split the corresponding 7-brane with $Q_{(7)} > 1$ are

[10] Note that the field theory on the 3-brane being IR free does not stabilize the ground state. The question of back reaction from 7-branes to 3-brane probes may be relevant in this context.

[11] Clearly, we are trying to be simplistic here. The intersection of 7-branes may require appropriate resolutions (or blowups), similar to the ones discussed [37]. Already in six dimensions some collisions did not have an interpretation in terms of conventional field theory and require tensionless strings. We expect similar phenomena in four-dimensional compactifications [38].

frozen. More precisely, only the splittings incompatible with the instanton embedding are forbidden. For example, a multiplicity two 7-brane with a $SU(2)$ instanton cannot split into two $U(1)$ 7-branes. Therefore on this branch of the moduli space the fourfold CY_4 is always singular.

Now let us return to the heterotic picture. When a 3-brane disappears from F-theory, a 5-brane should disappear from heterotic theory. This means that in that particular point on the Coulomb branch the 5-brane can be interpreted as a singular gauge field configuration such that the curvature is zero everywhere except on one fiber where it has a singularity. If (z, w) is a pair of local coordinates on the base so that $z = w = 0$ are the equations describing the fiber, the Pontryagin 4-form of the gauge field is $\mathrm{Tr} F \wedge F \sim k\delta^{(4)}(z, w, \bar{z}, \bar{w})$. The Higgs branch corresponds to smoothing out this singular configuration which changes the bundle $\mathcal{V}$ to a new bundle $\mathcal{W}$. In particular, $\int_{B_H} c_2(\mathcal{W}) = \int_{B_H} c_2(\mathcal{V}) + k$ so that $\mathcal{W}$ can take care of anomaly imbalance left when the 5-brane is removed. The way the new bundle $\mathcal{W}$ breaks the gauge symmetry should be the counterpart of the gauge symmetry breaking by the instanton field on the F-theory side.

Throughout this discussion we are making an assumption that the positions of 3-branes are independent from the positions of 7-branes inside B_F. This is essentially a version of the "probe argument" of [8]. If this is true, the 3-brane-instanton transition should not change the distribution of 7-branes, i. e. the complex structure of CY_4. It follows from the discussion in section 3 on F-theory – heterotic correspondence that the spectral curve Σ of the heterotic bundle $\mathcal{V}$ is preserved in the 5-brane-instanton transition in the dual picture: $\Sigma(\mathcal{W}) = \Sigma(\mathcal{V})$.

The only piece of spectral data that is left to change in the transition is the spectral bundle M, which lives on the component Σ of the spectral surface with $\mathrm{mult}(\Sigma) = n > 1$. It is clear that in general M has moduli (for sufficiently large $c_2(M)$, it does). This supports the idea that the multiple components of the spectral surface are in a one-to-one correspondence with the 7-branes carrying nonabelian gauge groups. To establish this correspondence at least in one case let us re-examine in the heterotic language the example described above in the F-theory language. Namely, we consider a 3-brane-instanton transition on the 7-brane which is wrapped around the zero section of $B_F \to B_H$ and carries the gauge group G. We can identify the 7-brane locus with the base B_H of the heterotic fibration $CY_3 \to B_H$.

The corresponding spectral surface on the heterotic side has several components. The nontrivial part of the $E_8 \times E_8$-bundle $\mathcal{V}$ is coded by the surface $\Sigma(\mathcal{V})$ and the line bundle $L(\mathcal{V})$. The unbroken gauge symmetry G corresponds to the component of the spectral surface which is the zero section S of the fibration $CY_3 \to B_H$. Again, the surface S can be identified with the base B_H. This component of the spectral surface carries a trivial G-bundle.

We start on the Coulomb branch so there are no instantons on the 7-brane and all the anomaly is cancelled by 3-branes (5-branes). Now let a 3-brane approach the 7-brane wrapped around the zero section. As we have discussed above, the Higgs branch develops

and the 3-brane dissolves into an instanton on B_H. Let us denote the corresponding background gauge bundle by $\tilde{M}$. In the heterotic picture, a 5-brane develops the Higgs branch and dissolves into an instanton so that the new heterotic bundle is $\mathcal{W}$. The bundle $\mathcal{W}$ has the same spectral surface $\Sigma(\mathcal{V})$ as the pre-transition bundle $\mathcal{V}$. However, on the zero section component of $\Sigma(\mathcal{V})$, a nontrivial spectral bundle M develops. The example discussed in section 3 shows that[12] actually $\mathcal{W} = \mathcal{V} \oplus \pi^* M$. The second Chern class $c_2(M) = k$ counts the 5-branes dissolved in the transition while $c_2(\tilde{M})$ counts the corresponding 3-branes. These numbers should be equal. Both M and $\tilde{M}$ were defined as bundles over B_H. Now we can describe the rest of the F theory-heterotic map: we suggest that simply $M = \tilde{M}$!

5. Matter in heterotic and F theories

5.1. Heterotic string after the 5-brane-instanton transition

In the previous section we discovered how the heterotic bundle $\mathcal{V}$ changes when several 5-branes dissolve to become instantons. The spectral surface of the resulting bundle $\mathcal{W}$ has a special form $\Sigma(\mathcal{W}) = \Sigma(\mathcal{V}) + nS$, i.e. it is a union of the spectral surface of $\mathcal{V}$ and the zero section nS. Multiplicity n of the zero section means e.g. that this component of the spectral surface carries a rank n bundle M. The zero section S is isomorphic to a base of elliptic fibration and therefore one can think about M as a vector bundle on a base B_H. Note that we have identified this bundle with the instanton bundle on the 7-brane. On the other hand the heterotic bundle $\mathcal{W}$ is (a deformation of) $\pi^* M \oplus \mathcal{V}$. The moduli of $\mathcal{W}$ are a part of the massless spectrum of the theory and thus are worth investigating.

To examine the deformations of a bundle of the form $\mathcal{W} = \mathcal{V} \oplus \pi^* M$ we look at the space

$$\begin{aligned} H^1(End(\mathcal{W})) = & H^1(End(\mathcal{V})) \oplus H^1(End(\pi^* M)) \oplus \\ & H^1(Hom(\mathcal{V}, \pi^* M)) \oplus H^1(Hom(\pi^* M, \mathcal{V})). \end{aligned} \tag{5.1}$$

The elements of $H^1(End(\mathcal{V})) \oplus H^1(End(\pi^* M))$ correspond to the deformations of $\mathcal{W}$ that preserve the direct sum decomposition structure and deform the two direct summands $\mathcal{V}$ and $\pi^* M$ independently. The elements of $H^1(Hom(\mathcal{V}, \pi^* M))$ give the deformations of $\mathcal{W}$ that are no longer direct sums but rather fit in an exact sequence. More precisely an element $\mu \in H^1(Hom(\mathcal{V}, \pi^* M))$ gives us an extension

$$0 \to \pi^* M \to \mathcal{W}_\mu \to \mathcal{V} \to 0 \tag{5.2}$$

which is a deformation of $\mathcal{W}$. Similarly an element $\nu \in H^1(Hom(\pi^* M, \mathcal{V}))$ corresponds to an extension

$$0 \to \mathcal{V} \to \mathcal{W}^\nu \to \pi^* M \to 0 \tag{5.3}$$

[12] More generally, $\mathcal{W}$ can be a deformation called a Hecke transform of $\mathcal{V} \oplus \pi^* M$ but for now the difference is not important.

which is another deformation of $\mathcal{W}$.

The decomposition (5.1) is true universally. On the *elliptic* CY_3 we can understand all three types of deformations in terms of the spectral data. That will allow us to find the counterparts of these moduli in F-theory. Let us start the discussion with the deformations of π^*M.

The deformations of π^*M are given by $H^1(CY_3, \pi^*End(M))$. Assume that M is a good instanton bundle on B_H so that $H^2(B_H, End(M)) = 0$. Then one can prove[13] that all the deformations of the pullback π^*M on CY_3 come from the deformations of M on the base B_H. This observation allows us to identify these moduli with the moduli $H^1(B_H, End(M))$ of the instanton background in F-theory. The number of such moduli is given by the index formula

$$-\chi(B_H, M) = 2nk - (n^2 - 1) \tag{5.4}$$

where k is the instanton number of M. We have already discussed this formula in the F-theory context.

Next, consider the deformations of the bundle $\mathcal{V}$ which is the heterotic bundle before the transition. The spectral data for $\mathcal{V}$ consist of the surface $\Sigma(\mathcal{V})$ and the line bundle $L(\mathcal{V})$. We assume that $\Sigma(\mathcal{V})$ has no multiple components so that any deformation $\beta \in H^1(CY_3, End(\mathcal{V}))$ is a deformation of the spectral surface $\Sigma(\mathcal{V})$. As discussed in section 3 in F-theory these deformations correspond to the complex structures of the Calabi-Yau fourfold CY_4.

Finally, the spectral surface of $\mathcal{W}$ has two components $\Sigma(\mathcal{V})$ and nS where $n = \text{rank}\,(M)$. Assume that M is well behaved (i.e., that $H^2(B_H, End(M)) = 0$) and let $\widetilde{\mathcal{W}} = \mathcal{V}^\beta \oplus \pi^*M^\alpha$ be a deformation of $\mathcal{W}$ having an infinitesimal $(\alpha, \beta) \in H^1(CY_3, End(\mathcal{V})) \oplus H^1(B_H, End(M))$. From the discussion above we see that the spectral surface of $\widetilde{\mathcal{W}}$ again has the form $\Sigma(\mathcal{V}^\beta) + nS$. Thus, the bundle M^α and the line bundle $L(\mathcal{V}^\beta)$ describe $\widetilde{\mathcal{W}}$ completely.

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[13] By the Lerray spectral sequence applied to the fibration $\pi : CY_3 \to B_H$ we have

$$\dim H^1(CY_3, \pi^*End(M)) = \dim H^1(B_H, \pi_*\pi^*End(M)) + \dim H^0(B_H, R^1\pi_*\pi^*End(M))$$

Now $\pi_*\pi^*End(M) = End(M) \otimes \pi_*\mathcal{O}_{CY_3}$ and $R^1\pi_*\pi^*End(M) = End(M) \otimes R^1\pi_*\mathcal{O}_{CY_3}$. Since the fibers of π are connected and compact we have $\pi_*\mathcal{O}_{CY_3} = \mathcal{O}_{B_H}$ and also relative duality gives $R^1\pi_*\mathcal{O}_{CY_3} = (\pi_*K_{CY_3/B_H})^* = (\pi_*(K_{CY_3} \otimes \pi^*K_{B_H}^{-1}))^* = (\pi_*(\pi^*K_{B_H}^{-1}))^* = (K_{B_H}^{-1} \otimes \pi_*\mathcal{O}_{CY_3})^* = K_{B_H}$. In other words there are two types of deformations of π^*M parameterized by the spaces $H^1(B_H, End(M))$ (deformations coming from the base) and $H^0(B_H, End(M) \otimes K_{B_H})$ (deformations that are non-trivial along the fibers). Since $End(M)$ is isomorphic to its own dual the space $H^0(B_H, End(M) \otimes K_{B_H})$ is dual to $H^2(B_H, End(M))$ and we conclude that all the deformations $\alpha \in H^1(CY_3, End(\pi^*M))$ of π^*M come from the base if and only if $H^2(B_H, End(M)) = 0$.

If, on the other hand, we have a deformation of the type, say, $\mathcal{W}_\mu$ with $\mu \in H^1(Hom(\mathcal{V}, \pi^*M))$, then the spectral surface of $\mathcal{W}_\mu$ is exactly the same as the spectral surface of $\mathcal{W}$. This is obvious since the exact sequence (5.2) guarantees that $\mathcal{W}$ and $\mathcal{W}_\mu$ have the same Harder-Narasimhan filtration when restricted on the general fiber of π. So what do the moduli $H^1(Hom(\mathcal{V}, \pi^*M))$ and $H^1(Hom(\pi^*M, \mathcal{V}))$ do with the spectral data?

To explain what is happening recall that a bundle $\mathcal{W}$ on CY_3 is encoded in a pair $(\Sigma(\mathcal{W}), L(\mathcal{W}))$ where $\Sigma(\mathcal{W}) \subset \check{CY}_3$ is a divisor (not necessarily reduced) and $L(\mathcal{W})$ is a line bundle on $CY_3 \times_{B_H} \Sigma(\mathcal{W})$. The bundle $\mathcal{W}$ is the push-forward of the sheaf $L(\mathcal{W})$ under the natural projection $CY_3 \times_{B_H} \Sigma(\mathcal{W}) \to CY_3$. One important feature of this description is that by definition the points of $\Sigma(\mathcal{W})$ over a point $b \in B_H$ are precisely the degree zero line bundles that participated in the associated graded of $\mathcal{W}_{|\pi^{-1}(b)}$ with respect to its Harder-Narasimhan filtration.

If it happens that $\Sigma(\mathcal{W})$ has two irreducible components $\Sigma(\mathcal{W}) = n'\Sigma' + n''\Sigma''$, then $\mathcal{W}$ comes furnished with extra structure. The restrictions L' and L'' of $L(\mathcal{W})$ on $n'\Sigma'$ and $n''\Sigma''$ respectively correspond say to vector bundles $\mathcal{V}'$ and $\mathcal{V}''$ of ranks $n'd'$ and $n''d''$, respectively. Here d' and d'' are the degrees of Σ' and Σ'' over B_H. The bundles $\mathcal{V}' \oplus \mathcal{V}''$ and $\mathcal{V}$ coincide outside the divisor $D := \pi^{-1}(\pi(\Sigma' \cap \Sigma''))$. More precisely $\mathcal{V}$ is a modification called a Hecke transform of $\mathcal{V}' \oplus \mathcal{V}''$ along D. Specifically this means that there is an exact sequence of sheaves on CY_3

$$0 \to \mathcal{V} \to \mathcal{V}' \oplus \mathcal{V}'' \to \mathcal{Q} \to 0 \tag{5.5}$$

where $\mathcal{Q}$ is the vector bundle on D obtained as a push-forward of the restriction of L' (or L'') on the intersection of the two divisors $CY_3 \times_{B_H} (n'\Sigma')$ and $CY_3 \times_{B_H} (n''\Sigma'')$ in $CY_3 \times_{B_H} \check{CY}_3$. The exact sequence (5.5) encodes the condition that L' and L'' come from a global line bundle on $\Sigma(\mathcal{V})$. There is one limit case when $\mathcal{V}$ itself becomes a direct sum. This happens precisely when $\mathcal{Q} = \mathcal{V}'_{|D} = \mathcal{V}''_{|D}$ and the map $\mathcal{V}' \oplus \mathcal{V}'' \to \mathcal{Q}$ is just the difference. In this case we have $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''(-D)$.

Let us now return to the special case $\mathcal{W} = \mathcal{V} \oplus \pi^*M$. The deformations of the type $\mathcal{W}_\mu$ (5.2) and $\mathcal{W}^\nu$ (5.3) with $\mu \in H^1(Hom(\mathcal{V}, \pi^*M))$ and $\nu \in H^1(Hom(\pi^*M, \mathcal{V}))$ do not deform the spectral surface $\Sigma(\mathcal{W})$. In fact these moduli can be identified with the moduli of all non-trivial Hecke transforms (5.5) we can perform on $\pi^*M \oplus \mathcal{V}(D)$ in order to obtain $\mathcal{W}_\mu$.

Certain information about the number of such moduli can be obtained by the index formula. If we denote $N = \dim H^1(Hom(\mathcal{V}, \pi^*M))$ and $\bar{N} = \dim H^1(Hom(\pi^*M, \mathcal{V})$ then from duality and Riemann-Roch it follows that

$$N - \bar{N} = \frac{1}{2}\mathrm{rank}(\mathcal{V})\; c_3(\pi^*M) - \frac{1}{2}n\; c_3(\mathcal{V}) = -\frac{1}{2}n\; c_3(\mathcal{V}). \tag{5.6}$$

Thus for $c_3(\mathcal{V}) \neq 0$ this matter is *chiral.* To find these moduli in F-theory we need to recall that matter can come from the intersections of D-branes. For instance, above

we have interpreted massless multiplets coming from intersections of the 7-brane with the dissolved 3-branes as the moduli $H^1(B_H, M)$ of the instanton background. We would like to relate the moduli $H^1(Hom(\mathcal{V}, \pi^*M))$ and $H^1(Hom(\pi^*M, \mathcal{V}))$ to multiplets coming from the intersections of the 7-brane wrapped on B_H with other 7-branes. To do that we would need to understand in F-theory the rôle of Hecke transform similar to (5.5).

5.2. Geometry of vector bundles on intersecting D branes

Let us return to the discussion started in section 4 about the matter fields in F-theory. We can classify them by the dimension of their support on the compact part of the 7-brane. Fields living in the bulk we call Type C. The ones living along curves on 7-branes we call Type B. In the simplest case these special curves are simply the intersections of two 7-branes. Finally, the fields coming from the points on the 7-branes are called Type A. The only example of Type A fields we will consider here are hypermultiplets which come from intersections with 3-branes.

We need to describe the vacua of the ABC system. The appropriate setup for this is given by the generalized Hitchin equations with the source terms. Without the source terms these equations describe the Type C fields in bulk. The source terms introduce couplings to Type A fields localized in points and to Type B fields localized along special curves within the 7-brane. To be concrete, we discuss 7-branes with SU gauge groups.

We start with the fields in the bulk. In 8 dimensions there is a vector multiplet. It has a complex scalar Ω in the adjoint. The nonzero *vev* of this field signals the splitting of the 7-brane to several parallel components. The eigenvalues of Ω measure the spatial separations of these components. After compactification of four dimensions on the complex surface X the scalar is twisted so it becomes a section of the canonical line bundle K_X or just a $(2,0)$-form Ω. The components of the eight-dimensional connection A_M along X determine the gauge background. Without coupling to Type A and Type B fields, the linearized equations of motion simply tell us that the background gauge bundle M is holomorphic and that Ω is a holomorphic section of $End(M) \otimes K_X$. That is, $\Omega \in H^2(X, End(M))$. If M is a well-behaved instanton bundle, the space $H^2(X, End(M))$ is empty, so necessarily $\Omega = 0$ which guarantees that the 7-brane carrying such an instanton bundle cannot split. This condition should be compared with the condition on the spectral bundle $\tilde{M}$ discussed in section 5.1 in the heterotic picture, which ensured that the multiple component of the spectral surface is preserved by the deformations of the bundle $\mathcal{W}$.

Now let us couple this system to Type A matter coming from a 3-branes with charge $Q_{(3)} = k$ sitting on top[14] of the 7-brane. Such 3-brane carries a IR trivial $N = 2$ theory with the gauge group $SU(k)$ and three multiplets in the adjoint. Its intersection with the 7-brane carrying $SU(N)$ gauge group produces two chiral multiplets Q and $\tilde{Q}$ in conjugate representations $(\mathbf{k}, \mathbf{N})$ and $(\bar{\mathbf{k}},\bar{\mathbf{N}})$ of $SU(k) \times SU(N)$. With respect to the $SU(N)$ gauge theory on the 7-brane the (decoupled) gauge group $SU(k)$ is a flavor symmetry.

[14] We assume that the 3-brane is away from the special divisors on the 7-brane as in [34].

To get a slightly different angle on the problem, let us start with the trivial gauge background M = id. Trivializing M we can think of the connection A_M just as of the holomorphic 1-form with values in the adjoint of $SU(N)$. Also, to the linear approximation, Ω is a holomorphic 2-form. So in the space-time there will be $h^{0,1}+h^{0,2}$ fields in the adjoint ($h^{0,1}$ fields h_a and $h^{0,2}$ fields Ω_i).

To get to the next approximation we notice that there is a topological coupling of two $(0,1)$ forms with one $(2,0)$ form [15]. In the effective $N=1$ space-time theory that gives a superpotential $W = C_{iab}\mathrm{Tr}(\Omega_i[h_a, h_b])$ with the coefficient C_{iab} determined by the intersection numbers in cohomology of X. The superpotential W also includes a term $\sum_i B^i \tilde{Q}\Omega_i Q$ where B_i are constants which couple Type A multiplets Q and $\tilde{Q}$ to Ω.

A theory with this superpotential can be easily analyzed. The classical vacua are determined by this superpotential together with the D-flatness condition and gauge invariance. The superpotential W is linear in Ω_i and bilinear in h_a and Q, $\tilde{Q}$. So the classical moduli space has a branch with

$$\Omega_i = 0, \quad B^i Q\tilde{Q} + \sum_{ab} C_i^{ab}[h_a, h_b] = 0, \quad i = 1, \ldots, h^{0,2}, \tag{5.7}$$

which has complex dimension $2Nk - (N^2-1)(1 - h^{0,1} + h^{0,2})$. In other words, this space is obtained starting from the complex vector space of dimension $2Nk + (N^2-1)h^{0,1}$ with coordinates given by the components of Q, $\tilde{Q}$ and h_a. In this space we take a complete intersection of $(N^2-1)h^{0,2}$ quadrics and finally divide by the action of the complexified gauge group $SL(N, \mathbf{C})$.

This space has the same dimension as the moduli space of $SU(N)$ instantons on X with the instanton number k. If $h^{0,2} = 0$ the moduli space of the Higgs branch is a symplectic quotient with respect to the action of $SU(N)$. When $h^{0,2} = 1$ (so that $X = T^4$ or $X = K3$) the construction above is a hyperkähler quotient. For $h^{0,2} > 1$ it is a generalization of the hyperkähler quotient. We conjecture that it gives a local description of the instanton moduli space of X in the vicinity of the point-like k-instanton configuration. Essentially this is a generalization of the ADHM construction on T^4 discussed in the similar context in [11].

Both Type A and Type C matter fields are non-chiral. Thus the only source of possible non-chirality is Type B matter which comes from intersections of pairs of 7-branes. Type B interacts with Type A and Type C via superpotentials which were qualitatively analyzed in section 4. For example, the superpotential $W(C, B)$ makes sure that the nonzero expectation values of Type B get translated into a nonzero expectation value of Ω. In turn, $\Omega \neq 0$ splits the 7-brane thus providing a D-brane realization of the Higgs mechanism.

[15] This coupling comes out from 8-dimensional coupling $\Omega\lambda\lambda$ under the dimensional reduction, where λ stands for twisted gaugino.

More concretely, suppose that $C \subset X$ is the curve where X intersects a 7-brane Y with $SU(K)$ gauge group. Type B matter multiplets q and $\tilde{q}$ transform respectively in $(\mathbf{N}, \mathbf{K})$ and $(\bar{\mathbf{N}}, \bar{\mathbf{K}})$ of $SU(N) \times SU(K)$. Assuming C is smooth, they can be computed as cohomology groups H^0 and H^1 of the bundle $M_X \otimes M_Y \otimes \mathcal{L}$ where $\mathcal{L}$ is a twisting line bundle on C. The chirality is measured by the Euler character of this bundle which is equal to $NK(1 - g(C) + \deg\mathcal{L})$. The latter formula should be compared, for $K = 1$ and $n = N$, with (5.6) in section 5.1. The 2-form Ω and the gauge connection A_M satisfy a set of equations with a δ-functional source term along C. These equations show in particular that Ω develops a pole along C with the residue $\tilde{q}q$ bilinear in the Type B matter fields. In a different context, the generalized Hitchin systems on curves instead of surfaces were also considered by N. Nekrasov in [39].

6. Discussion

Let us first summarize what we have already learned. The total moduli space of F-theory compactifications is a stratified space with numerous components. Each component is characterized by the brane configuration and the gauge bundles inside 7-branes. The components are connected through the transition points. The moduli space of the heterotic compactifications on Calabi-Yau threefolds is also stratified. Each component of the moduli space of the bundles on Calabi-Yau threefold is characterized by the number of components of the spectral surface and their multiplicities. The duality hypothesis implies that one can identify the corresponding strata of the F-theory moduli space with the corresponding strata of the heterotic compactification. The details of this identification depend on the strata.

The generic strata correspond to F-theory compactifications on *nonsingular* elliptic CY_4 (that means some inequalities for (n, m, k) that characterize the base of F-theory compactification). Then the elliptic fibration generically has only I_1 singularities. The gauge group is completely broken (except for some $U(1)$'s). Such compactifications are characterized by three integers (n, m, k) [14]. In the heterotic dual one has to specify two E_8 bundles characterized by the second Chern classes $c_2(V_1)$ and $c_2(V_2)$. The third Chern classes c_3 are identically equal to zero for E_8 bundles. The sum $c_2(V_1) + c_2(V_2)$ of the second Chern classes is fixed by the 5-brane anomaly cancellation condition, which leaves three independent integer-valued parameters, also.

When the elliptic fibration has higher singular loci (with singularity higher than I_1) the situation is different. We would like to propose the following picture: First let us consider the case when the discriminant of elliptic fibration has only one irreducible component (the 7-brane) with the higher singularity located on the zero section of $B_F \to B_H$ (see section 4). In order to specify the F-theory compactification, one has to fix the gauge bundle $\tilde{M}$ inside the 7-brane. This introduces two new integer-valued parameters — the rank r of the bundle and the second Chern class $c_2(\tilde{M})$. The total number of parameters that characterize this compactification is equal to 5. On the heterotic side one E_8 is

completely broken by a generic E_8 bundle while the other E_8 is broken by a vector bundle V. In general, the bundle V has the *nonzero* third Chern class $c_3(V)$. The rank and the structure group of the bundle V is fixed by the singularity type on F-theory side. The spectral surface $\Sigma(V)$ for this bundle is reducible and has a component with multiplicity r. This component is equipped with a vector bundle M of rank r with the second Chern class equal to $c_2(M) = c_2(\tilde{M})$. The third Chern class of the bundle V is related to $c_2(\tilde{M})$. Now, the independent parameters on the heterotic side are the three components of the second Chern class $c_2(V)$, the third Chern class $c_3(V)$ and the multiplicity r of one of the components of the spectral surface.

In general, both E_8 groups are not broken completely, so that that the residual gauge symmetry group is $G_1 \times G_2$. In the corresponding F-theory compactification the discriminant has (at least) two components (7-branes) with higher singularity located, say at the zero section and at the section at infinity of the bundle $B_F \to B_H$ (see section 4). The full set of data includes also the information about the background gauge bundles $\tilde{M}_i$ inside these two 7-branes — their ranks r_i and their second Chern classes $c_2(\tilde{M}_i)$. Again, the ranks of the bundles $\tilde{M}_i$ determine the multiplicities of various components of the spectral surface while the Chern classes $c_2(\tilde{M}_i)$ are related to the Chern classes of bundles V_i. The parameters that characterize the F-theory compactification are n, m, k, the Chern classes $c_2(\tilde{M}_i)$ and the ranks r_i of the bundles inside the 7-branes. These parameters match with the corresponding parameters on the heterotic side — the components of the second Chern classes $c_2(V_i)$, the third Chern classes $c_3(V_i)$ and the multiplicities of components of the spectral surface.

We want to emphasize the striking analogy between the D-branes in the F-theory description and the spectral surfaces (curves) in the heterotic description. The complex deformations of the spectral surface match the complex deformations of the collection of 7-branes. Similarly, the background gauge fields inside the 7-branes map to the bundle on the spectral surface. The gauge symmetry enhancement mechanisms are very similar and the transition of a 5-brane into a finite-size instanton is very similar to the instanton — 3-brane transition.

There is also a similarity in how the matter multiplets appear in both theories. In F-theory one expects chiral matter to be produced on the intersections of 7-branes. On the heterotic side the chiral matter multiplets can be expressed as cohomological groups localized to the intersections of spectral surfaces. Another source of (nonchiral) matter in F-theory is provided by the open strings connecting 7- and 3-branes. In the heterotic description, the corresponding multiplets are related (see section 5) to the moduli $H^1(Hom(\pi^* M, \mathcal{V}))$ responsible for smoothing out the pointlike instantons (5-branes) into the finite-size instantons. It would be very interesting to understand if there is any *physical* meaning to the analogy between D-branes and spectral surfaces, beyond the apparent similarity in the math apparatus.

Another interesting question we only lightly touch upon in section 3 of this paper is the appearance of tensionless strings [40][41][18] that should play an important role in our

understanding of various nonperturbative phenomena. Generically, in four-dimensional compactifications the 7-branes intersect each other over curves where the singular fiber jumps. The Calabi-Yau fourfold may require a resolution. In some cases it is not enough to blow up the singular fiber and one also needs to blow up the base. This leads to a variety of phase transitions.

Clearly, in this paper we just have begun to explore the four-dimensional F-theory compactifications. There are plenty of important questions still open, such as the detailed structure of the map between F-theory — heterotic moduli and the clear understanding of matter spectrum. All intricate phenomena known in $N = 1$ four-dimensional field theories should be derivable from F-theory. One of the real challenges is to understand the famous Seiberg's duality [42]. The first steps in this direction were done in [21][22].

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8. Appendix

Here we list (without proofs) some facts that are used throughout the paper.

8.1. Hirzebruch surface $\mathbf{F}_n$ and elliptic Calabi-Yau

Hirzebruch surface is a $\mathbf{P}^1$ bundle over $\mathbf{P}^1$. One can think about it as a toric variety. Let z, w, u and v be coordinates in C^4. Define the action of two $U(1)$s, given as follows $\lambda : (z, w, u, v) \to (\lambda z, \lambda w, \lambda^n u, v)$ and $\mu : (z, w, u, v) \to (z, w, \mu u, \mu v)$. Then the Hirzebruch surface $\mathbf{F}_n$ is defined as

$$\big(C^4 \setminus \{\text{fixed set}\}\big)/(\lambda, \mu) \ . \tag{8.1}$$

$H^2(\mathbf{F}_n)$ of Hirzerbruch surface is generated by the zero section $\tilde{b}$ and a fiber $\tilde{a}$. The intersection pairing of these elements is $\tilde{a}^2 = 0, \tilde{b}^2 = -n$ and $\tilde{a}\tilde{b} = 1$.

Consider the nonsingular elliptic Calabi-Yau threefold fibered over $\mathbf{F}_n$. The fourth cohomology $H^4(CY_3)$ is three-dimensional and is generated by A, B, S. We assume that $\pi(A) = \tilde{a}, \pi(B) = \tilde{b}$ and $\pi(S) = \mathbf{F}_n$. The triple intersections are equal to

$$AS^2 = -2, \quad BS^2 = -2 + n, \quad B^2S = -n, \quad ABS = 1, \quad S^3 = 8 \ , \tag{8.2}$$

all other triple intersections are equal to zero.

8.2. Generalized Hirzebruch $\mathbf{F}_{nmk}$

For simplicity we choose the base being the $\mathbf{P}^1$ bundle over the Hirzebruch surface $\mathbf{F}_n$ (generalized 3-dimensional Hirzebruch $\mathbf{F}_{nmk}$). Let (z, w, u, v, t, s) be the coordinates in C^6. Define three $U(1)$ actions as follows

$$\begin{aligned} \lambda &: (z, w, u, v, s, t) \to (\lambda z, \lambda w, \lambda^n u, v, \lambda^m s, t) \\ \mu &: (z, w, u, v, s, t) \to (z, w, \mu u, \mu v, \mu^k s, t) \\ \nu &: (z, w, u, v, s, t) \to (z, w, u, v, \nu s, \nu t) \ . \end{aligned} \tag{8.3}$$

Then the generalized three-dimensional Hirzebruch is defined as the quotient

$$\left(C^6 \setminus \{\text{fixed set}\}\right)/(\lambda, \mu, \nu) \tag{8.4}$$

For future applications we present here the intersection ring of $\mathbf{F}_{nmk}$. The ring is generated by three elements a, b, c satisfying the following relations

$$a^2 = 0, \quad b^2 = -nab, \quad c^2 = -kbc - mac \ . \tag{8.5}$$

The nonzero intersection pairings are

$$abc = 1, \quad b^2c = -n, \quad c^3 = 2km - k^2n, \quad c^2b = kn - m, \quad c^2a = -k \ . \tag{8.6}$$

In the case of *smooth* CY_4 one can immediately compute the Euler character in terms of some classes of the base [5]

$$\frac{1}{24}\chi = \int 15c_1^3 + \frac{1}{2}c_1c_2 = 732 + 60km - 30k^2n \ . \tag{8.7}$$

In the smooth case without any gauge field inside the 7-branes (trivial bundle), this number counts the 3-brane (5-brane) anomaly. In the case when the elliptic fibration has singularities higher than I_1, the Euler character can be computed using the methods of [23].

8.3. Vector bundles on elliptic fibrations

Elliptic fibrations. An elliptic fibration is a fibration $\pi : X \to B$, where X and B are smooth projective varieties and the f is a flat morphism whose fibers are connected curves of arithmetic genus one. Unless stated otherwise we will assume that the singular fibers of π are always reduced and have at most ordinary double points as singularities. Also we will require that the fibration $\pi : X \to B$ possess a section $\sigma : B \to X$. In this case X has a natural structure of an abelian group scheme over B and σ is the neutral element in the Mordell-Weyl group (= the group of global sections).

Denote by $\check{\pi} : \check{X} := \mathrm{Pic}^0(X/B) \to B$ the degree zero relative Picard of π. The general fibers of $\check{\pi}$ are just the elliptic curves dual to the corresponding fibers of π. The existence of σ guarantees that as an elliptic fibration $\check{\pi} : \check{X} \to B$ is isomorphic to $\pi : X \to B$. However, we will keep distinguishing X and $\check{X}$ for the time being so that we can trace the sources of the different geometric objects in our construction.

For computational purposes it is convenient to think of the fibration $\pi : X \to B$ in terms of its Weierstrass model, which we proceed to describe. Put $K_{X/B}$ for the relative canonical bundle of π. Its push-forward $\alpha := \pi_* K_{X/B}$ is a line bundle on B and X sits naturally as a divisor in $\mathbf{P}(\mathcal{O}_B \oplus \alpha^{\otimes 2} \oplus \alpha^{\otimes 3})$. Explicitly, there exist sections $f \in \Gamma(B, \alpha^{\otimes 4})$ and $g \in \Gamma(B, \alpha^{\otimes 6})$ so that the affine piece of X sitting in the total space of the vector bundle $\alpha^{\otimes 2} \oplus \alpha^{\otimes 3}$ is given by the equation

$$y^2 = x^3 + a^* f x + a^* g. \tag{8.8}$$

Here $a : \mathbf{P}(\mathcal{O}_B \oplus \alpha^{\otimes 2} \oplus \alpha^{\otimes 3}) \to B$ is the natural projection and x and y are the tautological sections of the pullbacks of $\alpha^{\otimes 2}$ and $\alpha^{\otimes 3}$, respectively [43]. The discriminant locus of π is the divisor of the section $\Delta := 4f^3 + 27g^2 \in \Gamma(B, \alpha^{\otimes 12})$. For future reference notice that since $\pi_* K_X = \alpha \otimes K_B$, the variety X will have a trivial canonical bundle if and only if $\alpha = K_B^{-1}$.

Vector bundles. We will be interested in instanton bundles with vanishing first Chern class on elliptic fibrations. Notice that if $V \to X$ is such a bundle, then the restriction of V to any smooth fiber of π is a direct sum of indecomposable vector bundles of degree zero. An indecomposable vector bundle of degree zero on an elliptic curve is completely determined by its rank r and by a line bundle γ of degree zero. More precisely, every such bundle is of the form $E_r \otimes \gamma$ where E_r is the unique indecomposable vector bundle of rank r and degree 0 for which the associated graded of the Harder-Narasimhan filtration is a direct sum of r copies of the trivial line bundle $\mathcal{O}$ [44]. It is easy to see that the restriction of a general V to the general fiber of π will be a direct sum of $\mathrm{rank}(V)$ line bundles of degree zero. The collection of these line bundles can be viewed as a collection of points on the dual elliptic curve and by varying everything over the base we obtain from V a subscheme $\Sigma(V)$ in $\check{X}$ mapping generically finitely to B with degree $\mathrm{rank}(V)$. This subscheme encodes some part of the geometric information contained in V but is not sufficient for the reconstruction of V. To recover the missing piece of the puzzle let us examine more closely the case when the map $\pi : X \to B$ is smooth. One has the following

Proposition 1. *Let $\pi : X \to B$ be an elliptic fibration without singular fibers. The following objects are equivalent*

(i) *A rank r vector bundle V on X with* $\det V = \mathcal{O}_X$*;*

(ii) *A pair (Σ, L) where $\Sigma \subset \check{X}$ is a subscheme for which $\check{\pi} : \Sigma \to B$ is finite of degree r, and L is a rank one sheaf on Σ;*

Informally we can pass from (i) to (ii) as follows. Take a point $t \in B$ and let $X_t = \pi^{-1}(t)$ be the elliptic curve in X sitting over t and $V_t = V_{|X_t}$ be the restriction of V to X_t. As we explained above the bundle V_t is a direct sum $V_t = E_{k_i} \otimes \alpha_i$ with $\alpha_i \in \mathrm{Pic}^0(X_t)$. The points of Σ sitting over t are the points $\alpha_i \in \check{X}_t$ (α_i counts with multiplicity k_i) and the fiber of L at the point α_i is the vector space $H^0(X_t, \mathrm{Hom}(E_{k_i} \otimes \alpha_i, V_t))$ of all global maps between $E_{k_i} \otimes \alpha_i$ and V_t. Note that in general $\alpha_i \neq \alpha_j$ and hence $H^0(X_t, \mathrm{Hom}(E_{k_i} \otimes \alpha_i, V_t))$ is a one-dimensional vector space.

To describe the correspondence of data of type (i) and type (ii) more rigorously, consider the fiber product $X \times_B \check{X}$ with the two natural projections $p : X \times_B \check{X} \to X$ and $\check{p} : X \times_B \check{X} \to \check{X}$. Denote by $\mathcal{P}$ the relative Poincare bundle on $X \times_B \check{X}$ normalized so that the pullback of $\mathcal{P}$ to X via the zero section $\check{\sigma} : X \to X \times_B \check{X}$ is $\mathcal{O}_X$ and the pullback of $\mathcal{P}$ to $\check{X}$ via the zero section $\sigma : \check{X} \to X \times_B \check{X}$ is $\mathcal{O}_{\check{X}}$. Recall that $\mathcal{P}$ is uniquely characterized by the normalization condition and by the property: for every $t \in B$ and every $\alpha \in \check{X}_t = \check{p}^{-1}(t) = \mathrm{Pic}^0(X_t)$ there is an isomorphism $\mathcal{P}_{|X_t \times \{\alpha\}} \cong \alpha$. Now we are ready to formalize the passage between the data (i) and (ii).

Starting with a vector bundle V on X with trivial determinant, we can form the sheaf $\mathcal{F}(V) := \check{p}_*(p^*V \otimes \mathcal{P}^{-1})$ on $\check{X}$. By construction $\mathcal{F}(V)$ is a torsion sheaf on $\check{X}$ supported at the set of points $\alpha \in \check{X}$ that have the property $\dim H^0(X_{\check{\pi}(\alpha)}, \alpha^{-1} \otimes V_{\check{\pi}(\alpha)}) \neq 0$. In particular the support $\Sigma(V)$ of $\mathcal{F}(V)$ is a divisor in $\check{X}$ that maps $r : 1$ to the base B. Alternatively $\mathcal{F}(V)$ can be thought of as the extension by zero of a sheaf $L(V)$ on $\Sigma(V)$ and it is not hard to check that $L(V)$ must have rank one. Conversely, if we start with a pair (Σ, L) we can construct a vector bundle $V(\Sigma, L)$ of rank r on X in the following way: The fiber product $Y := X \times_B \Sigma$ is a smooth elliptic fibration over Σ via the natural projection $p_\Sigma : Y \to \Sigma$. Put $p_X : Y \to X$ for the projection on X and define $V(\Sigma, L) = p_{X*}(p_\Sigma^* L \otimes \mathcal{P} \otimes \omega_{Y/X}^{-1})$ where as before $\mathcal{P}$ is the (restriction of) the Poincare bundle from $X \times_B \check{X}$ and $\omega_{Y/X}$ is the relative dualizing sheaf of the map $p_X : Y \to X$. It is not hard to convince oneself that the two assignments $V \mapsto (\Sigma(V), L(V))$ and $(\Sigma, L) \mapsto V(\Sigma, L)$ are inverse to each other. Also note, that in general position Σ will be a smooth cover and L will be a line bundle on Σ. In that case Y is also smooth and by the Hurwitz formula $\omega_{Y/X}^{-1} = \mathcal{O}_Y(-R)$, with $R \subset Y$ being the ramification divisor of the projection $p_X : Y \to X$.

When we allow singular fibers in the elliptic fibration $\pi : X \to B$ the above simple correspondence between the data (i) and (ii) does not hold literally even when we are in general position. It turns out that the smoothness of Σ does not in general imply the smoothness of Y and that it is necessary to modify the assignment $(\Sigma, L) \mapsto V(\Sigma, L)$ along the singular locus of Y. Instead of discussing the necessary modifications in full generality, we will briefly explain below what needs to be done in the specific situations when X is $K3$ surface or a Calabi-Yau 3-fold.

Among other things this description of vector bundles on X leads to a peculiar compactification of the moduli space which is obtained as follows: Fix a set of cohomology classes

$\underline{c} \in \oplus_{i\geq 2}(H^{i,i}(X) \cap H^{2i}(X, \mathbf{Z}))$ on X. Fix an ample line bundle H on X and denote by $M_X(r, \underline{c})$ the moduli space of rank r bundles on X with Chern classes $\underline{c}$ that are Gieseker semistable with respect to H. The polarization H induces a canonical polarization $\tilde{H}$ on the fiber product $X \times_B \check{X}$ and the H-stability condition on a bundle V is equivalent to the $\tilde{H}$-stability of L considered as a torsion sheaf on $X \times_B \check{X}$ supported on the divisor $X \times_B \Sigma$. The Hilbert polynomial p of this torsion sheaf can be calculated entirely in terms of $\underline{c}$ and so we can identify $M_X(r, \underline{c})$ with a Zariski open subset in the moduli of sheaves $M_{X\times_B \check{X}}(p)$. The structure of the latter is rather simple. The morphism assigning to a sheaf its support realizes $M_{X\times_B \check{X}}(p)$ as a fibration over the set of all Σ's whose fibers are compactified Picard varieties. It is easy to see that for a fixed $\underline{c}$ the various divisors Σ are all linearly equivalent and so the base of this fibration is a projective space.

8.4. The $K3$ case

Let us examine the case when X is a $K3$ surface in more details. In this case the base B is the projective line. We will assume that X is generic in the sense that π has exactly 24 singular fibers. This situation has the advantage that at least for a general V the branch points of $\Sigma(V)$ will be different from the discriminant of $\pi : X \to B$ and so the fiber product $Y = X \times_B \Sigma(V)$ will be smooth. This allows us to use the above procedure for passing between V and (Σ, L) without any further modifications. Let S_X, F_X be the classes of the zero section and the fiber, respectively. It is known that $F_X^2 = 0$, $F_X S_X = 1$ and $S_X^2 = -2$ and that for a general such X we have $\mathrm{Pic}(X) = \mathbf{Z}S_X \oplus \mathbf{Z}F_X$. Similarly we have $\mathrm{Pic}(\check{X}) = \mathbf{Z}S_{\check{X}} \oplus \mathbf{Z}F_{\check{X}}$.

Using these bases it is not hard to express the numerical invariants of (Σ, L) in terms of the numerical invariants of V. If V is a rank r vector bundle on X with trivial first Chern class, then by construction $\Sigma(V) = rS_{\check{X}} + kF_{\check{V}}$. To find the coefficient k consider the intersection $S_{\check{X}} \cdot \Sigma(V)$. For V in general position, $S_{\check{X}} \cdot \Sigma(V)$ consists of $k - 2r$ distinct points[16] on $S_{\check{X}} \subset \check{X}$. A point $\alpha \in S_{\check{X}} \cap \Sigma(V)$ corresponds to a copy of the trivial line bundle appearing as a direct summand in $V_{\check{\pi}(\alpha)}$. Therefore $k - 2r = \dim H^0(B, \pi_* V) = \dim H^0(B, R^1\pi_* V)$. Furthermore, the fact that $\pi_* V$ is a torsion sheaf on the curve B and the Lerray-Serre spectral sequence imply that $\dim H^1(X, V) = \dim H^0(B, R^1\pi_* V) = k - 2r$. If V is stable on X, then $H^0(X, V) = 0$ and by duality $H^2(X, V) = 0$. Thus $k - 2r = \dim H^1(X, V) = -\chi(X, V)$ which can be calculated by the Hirzebruch-Riemann-Roch formula. We have

$$2r - k = \chi(X, V) = [ch(V)td(T_X)]_2 = [(r - c_2(V)t^2)(1 + 2t^2)]_2 = 2r - c_2(V),$$

and so $k = c_2(V)$. To calculate the degree of the line bundle $L(V)$ in terms of V we just have to notice that $\sigma^*(V)$ is going to be the pushforward of the line bundle on $\Sigma(V)$

[16] Notice that since $S_{\check{X}}$ and $\Sigma(V)$ are both effective, $k > 2r$ is a necessary condition for the existence of the bundle V.

which is the restriction of $p_\Sigma^* L(V) \otimes \mathcal{P} \otimes \omega_{Y/X}^{-1}$ to the zero section of the elliptic fibration $p_\Sigma : Y \to \Sigma$. By the normalization condition on the Poincare bundle we know that $\mathcal{P}$ restricts to the trivial line bundle on this section. Also, since Y is the fiber product of Σ and X, it follows that the ramification divisor of the covering $p_X : Y \to X$ is the pull-back of the ramification divisor of $\check{\pi} : \Sigma(V) \to B$, which, combined with the condition $\deg \sigma^*(V) = 0$, yields $\deg L(V) = -1/2 \deg K_{\Sigma(V)/B}$. To summarize:

Claim 1. *Let $V \to X$ be a rank r vector bundle with $\det(V) = \mathcal{O}_X$. Let (Σ, L) be the pair corresponding to V. Then $\Sigma = rS_{\check{X}} + c_2(V)F_{\check{X}}$ and $\deg L = -(r+g-1)$ where $g = g(\Sigma) = rc_2(V) - r^2 + 1$.*

In these terms it is easy to describe the moduli space $M_X(r,k)$ of rank r vector bundles on X with trivial first Chern class and second Chern number k. Fix a smooth $\Sigma \subset \check{X}$ which is linearly equivalent to $rS_{\check{X}} + kF_{\check{X}}$. Let $M_\Sigma \subset M_X(r,k)$ be the subvariety, parameterizing vector bundles giving rise to Σ. The fibration $p_\Sigma : Y \to \Sigma$ has two natural sections S_Y and T_Y. The section S_Y is the pull-back of S_X via p_X and T_Y is the graph of the embedding $\Sigma \subset X$ obtained from the identification $X \cong \check{X}$. It is straightforward to calculate the intersections of S_Y and T_Y on Y. We have $S_Y^2 = -2r, T_Y^2 = -2r, S_Y T_Y = k - 2r$. Also for a general Y one has $\mathrm{Pic}(Y) = \mathrm{Pic}(\Sigma) \oplus \mathbf{Z}S_Y \oplus \mathbf{Z}T_Y$. For the Poincare bundle $\mathcal{P}$ one calculates $\mathcal{P} = \mathcal{O}_Y(S_Y - T_Y) \otimes p_\Sigma^* \rho$, where ρ is line bundle on Σ of degree $-c_2(V)$. This combined with Proposition 1 gives

Claim 2. *Let $d = c_2(V) + r + g - 1$. The natural map $\varphi_\Sigma : \mathrm{Pic}^d(\Sigma) \to M_\Sigma$ given by $\xi \mapsto p_{X*}(p_\Sigma^* \xi \otimes \mathcal{O}_Y(S_Y - T_Y))$ is an isomorphism.*

As a corollary we immediately obtain

Corollary 1. *$M_X(r,k)$ is birationally isomorphic to the total space of the family of Jacobians of degree $k + r + g - 1$ (equivalently $-(r+g-1)$) of the curves in the linear system $\mid rS_{\check{X}} + kF_{\check{X}} \mid$. In particular the smooth part of $M_X(r,k)$ is a hyperkähler manifold which is also a completely integrable Hamiltonian system.*

8.5. The Calabi-Yau case

Suppose now that $B \cong \mathbf{F}_n$ is a Hirzebruch surface and that X is a three-dimensional Calabi-Yau manifold. We know that $\mathrm{Pic}(B) = \mathbf{Z}\tilde{a} \oplus \mathbf{Z}\tilde{b}$, where $\tilde{a}$ is the fiber of the Hirzebruch surface B and $\tilde{b}$ is the infinity section. We have $\tilde{a}^2 = 0$, $\tilde{b}^2 = -n$ and $\tilde{a}\tilde{b} = 1$. The Chow ring of a generic X of this type is generated by the three divisor classes $A_X := \pi^*\tilde{a}$, $B_X := \pi^*\tilde{b}$ and $S_X = \sigma(B)$, with relations $A_X^2 = B_X^2 = 0$ and the ones given by the formulas (8.2). In particular the Picard group of X is freely generated by A_X, B_X, S_X as an abelian group and the integral part of $H^{2,2}(X)$ is freely generated by the curves $A_X B_X$, $B_X S_X$ and $A_X S_X$. Similarly we have classes $A_{\check{X}}$, $B_{\check{X}}$ and $S_{\check{X}}$ for $\check{X}$. Thus we can write $c_2(V) = c_2(V)_{AB} A_X B_X + c_2(V)_{AS} A_X S_X + c_2(V)_{BS} B_X S_X$ for any vector bundle V on X. As in the construction discussed in Proposition 1 we can form

the torsion sheaf $\check{p}_*(p^*V \otimes \mathcal{P}^{-1})$. Its support $\Sigma(V)$ will be a surface in $\check{X}$ for which the map $\check{\pi} : \Sigma(V) \to B$ is generically finite of degree r. To recover the numerical invariants of $(\Sigma(V), L(V))$ in terms of those of V, we just have to notice that the general members of the linear systems $|A_X|$ and $|B_X + n/2A_X|$ are smooth elliptic $K3$ surfaces sitting in X. After restricting to those and applying what we already know about the $K3$ case we obtain $\Sigma(V) = rS_{\check{X}} + c_2(V)_{AS}A_{\check{X}} + c_2(V)_{BS}B_{\check{X}}$. The information about the component $c_2(V)_{AB}$ of $c_2(V)$ can also be read off from the pair (Σ, L). Indeed, since $c_2(V)_{AS}$ and $c_2(V)_{BS}$ are determined by Σ, it suffices to compute the intersection $c_2(V) \cdot S_X = c_2(V_{|S_X})$ in terms of L and Σ. On the other hand by construction we have $V_{|S_X} = \pi_*(L \otimes \omega^{-1}_{\Sigma/B})$. Put $M := L \otimes \omega^{-1}_{\Sigma/B}$. The Grothendieck-Riemann-Roch formula for the finite map $\pi : \Sigma \to B$ reads

$$ch(\pi_* M) td(T_B) = \pi_*(ch(M) td(T_\Sigma))$$

and in combination with the condition $c_1(V) = 0$ and the Riemann-Roch theorem for the line bundle M on the surface Σ this yields

$$c_2(V) \cdot S_X = r \cdot td_2(T_B) + (K_\Sigma \cdot M - M^2)/2 - td_2(T_\Sigma) = r \cdot td_2(T_B) - \chi(M) + \chi(\mathcal{O}_\Sigma) - td_2(T_\Sigma).$$

It is also straightforward to check that for the Hirzebruch surface B one has $td_2(T_B) = 1$. Since the Hirzebruch-Riemann-Roch formula on Σ gives $\chi(\Sigma, \mathcal{O}_\Sigma) = td_2(T_\Sigma)$, we get summarily the following

Claim 3. *Let $V \to X$ be a rank r vector bundle with* $\det(V) = \mathcal{O}_X$. *Let $(\Sigma(V), L(V))$ be the pair corresponding to V. Then $\Sigma(V) = rS_{\check{X}} + c_2(V)_{AS}A_{\check{X}} + c_2(V)_{BS}B_{\check{X}}$ in* $\mathrm{Pic}(\check{X})$ *and $c_2(V) \cdot S_X = r - \chi(\Sigma(V), L(V) \otimes \omega^{-1}_{\Sigma(V)/B})$.*

In order to recover the bundle V from the pair $(\Sigma(V), L(V))$, we have to modify slightly the construction from Proposition 1. The modification is forced by the fact that even when $\Sigma(V)$ is smooth the fibered product $X \times_B \Sigma(V)$ will be singular since the branch divisor of $\Sigma(V) \to B$ will always intersect the discriminant of $\pi : X \to B$, which is ample. The singularities of $X \times_B \Sigma(V)$ occur at the intersection points of the branch and the discriminant divisors and are therefore isolated. If $\nu : Y \to X \times_B \Sigma(V)$ denotes a resolution of these singularities and $p_\Sigma : Y \to \Sigma(V)$ and $p_X : Y \to X$ are the natural projections, one can check that V is isomorphic to the push-forward $p_{X*}\mathcal{L}(V)$ of a suitable rank one sheaf $\mathcal{L}(V) \to Y$. The sheaf $\mathcal{L}(V)$ can be reconstructed from $L(V)$ as $\mathcal{L} = p_\Sigma^* L(V) \otimes \nu^*\mathcal{P} \otimes \omega^{-1}_{Y/X} \otimes \mathcal{O}_Y(\ell E)$, where $E \subset Y$ is the exceptional divisor of ν. The integer ℓ is completely determined by (and determines) the third Chern class of V.

It can be checked that the condition that the linear system $|rS_{\check{X}} + c_2(V)_{AS}A_{\check{X}} + c_2(V)_{BS}B_{\check{X}}|$ contains an effective divisor, implies that the line bundle $\mathcal{O}_{\check{X}}(rS_{\check{X}} + c_2(V)_{AS}A_{\check{X}} + c_2(V)_{BS}B_{\check{X}})$ is actually ample on $\check{X}$. By Bertini's theorem the general spectral surface Σ will be smooth and connected. Moreover for such a surface the Lefschetz hyperplane section theorem gives $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$ and $H^{1,1}(\check{X}) \subset H^{1,1}(\Sigma)$. Let $\underline{c}$ be

the Chern classes of V. For a fixed $(\Sigma, \mathcal{L})$ the Künneth formula applied to $X \times_B \Sigma$ shows that $\mathcal{L}$ and L have the same number of moduli. Therefore the support map $M_X(r, \underline{c}) \to |rS_{\check{X}} + c_2(V)_{AS}A_{\check{X}} + c_2(V)_{BS}B_{\check{X}}|$ is surjective and generically finite.

Remarks.

1. In contrast with the $K3$ case, the moduli space $M_X(r, \underline{c})$ may be reducible and may have components of different dimension. It can also happen that the support map contracts whole components of $M_X(r, \underline{c})$. Examples like that can be easily constructed by taking direct sums of bundles on X with pull-backs of bundles on B.

2. By degenerating X to a double generalized Hirzebruch, it can be shown that for the general pair (X, Σ) the only divisor classes on Σ are the restrictions of $A_{\check{X}}$, $B_{\check{X}}$ and $S_{\check{X}}$. In particular $\mathcal{L}$ is a linear combination of the exceptional divisor E and of the strict transforms of the zero section of $X \times_B \Sigma \to \Sigma$, the divisor $\mathcal{T} \subset X \times_B \Sigma$ corresponding to the graph of the embedding $\Sigma \subset X$, and the pull-backs of $A_{\check{X}|\Sigma}$, $B_{\check{X}|\Sigma}$ and $S_{\check{X}|\Sigma}$ to $X \times_B \Sigma$. The six coefficients of $\mathcal{L}$ in this basis are not independent. There are relations between them coming from the identification $p_{X*}\mathcal{L} = V$ and the condition $c_1(V) = 0$ and from fixing $c_3(V)$ and $c_2(V) \cdot S_X$.

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